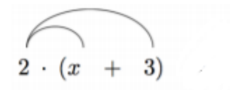
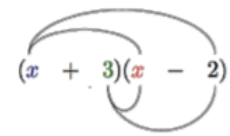
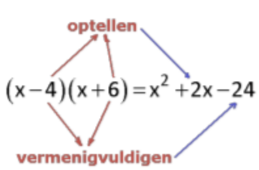
Wiskunde B refresher 1

- Bewerkingsvolgorde
  - Haakjes ( )
  - Machtsverheffen en worteltrekken $^2\sqrt{\ }$
  - Vermenigvuldigen en delen $\times\ :$
  - Optellen en aftrekken $+-$
    - Gelijkwaardig: van links naar rechts
- Haakjes wegwerken
  - Enkele haakjes

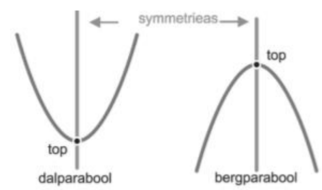


  - Dubbele haakjes

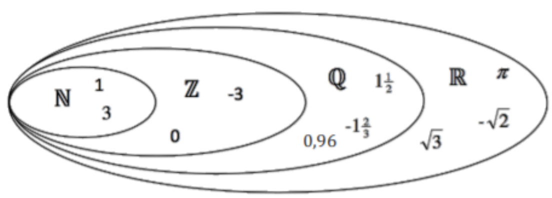




- $f(x) = ax^2 + bx + c \quad (a \neq 0)$
- Grafiek is een parabool
  - $a > 0$: dalparabool
  - $a < 0$: bergparabool



$\mathbb{N}$	Natuurlijke getallen	Positieve gehele getallen: $\{0, 1, 2, 3, \ldots\}$
$\mathbb{Z}$	Gehele getallen (integers)	Positieve en negatieve gehele getallen: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	Rationale getallen	Getallen die te schrijven zijn als $\frac{a}{b}$ waarbij $a \in \mathbb{Z} \wedge b \in \mathbb{N}_{>0}$
$\mathbb{R}$	Reële getallen	Alle rationale en irrationale getallen



Typen tweedegraadsvergelijkingen	
Twee termen	
$\boldsymbol{ax^2 + bx = 0}$ Haal x buiten haakjes. $3x^2 - 7x = 0$ $x(3x - 7) = 0$ $x = 0 \vee 3x = 7$ $x = 0 \vee x = \frac{7}{3} = 2\frac{1}{3}$	$\boldsymbol{ax^2 + c = 0}$ Herleid tot de vorm $x^2 =$ getal. $3x^2 - 75 = 0$ $3x^2 = 75$ $x^2 = 25$ $x = 5 \vee x = -5$
Drie termen	
Het linkerlid is te ontbinden Ontbind het linkerlid. $x^2 - 5x - 14 = 0$ $(x + 2)(x - 7) = 0$ $x = -2 \vee x = 7$	**Het linkerlid is niet te ontbinden** Gebruik de *abc*-formule. $3x^2 - 2x - 2 = 0$ De discriminant is $D = (-2)^2 - 4 \cdot 3 \cdot -2 = 28$ $x = \frac{2 - \sqrt{28}}{6} \vee x = \frac{2 + \sqrt{28}}{6}$

Aanpak abc-formule

1. Herleid het rechterlid tot 0
   - $ax^2 + bx + c = 0$
2. Bereken de discriminant $D = b^2 - 4ac$
   - $D < 0$: geen oplossingen
   - $D = 0$: één oplossingen
   - $D > 0$: twee oplossingen
3. De oplossingen zijn $x = \frac{-b - \sqrt{D}}{2a}$ en $x = \frac{-b + \sqrt{D}}{2a}$

- Rekenregels
  - $\sqrt{A \cdot B} = \sqrt{A} \cdot \sqrt{B}$ en $\sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}}$
    - Kijk uit dat $\sqrt{A + B} \neq \sqrt{A} + \sqrt{B}$

- Afspraken
  1. Zet onder het wortelteken een zo klein mogelijk geheel getal
     - $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16} \cdot \sqrt{5} = 4\sqrt{5}$
  2. Laat geen breuk onder het wortelteken staan
     - $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$
  3. Laat nooit een wortel in de noemer van een breuk staan
     - $\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} = \frac{5}{2}\sqrt{2} = 2\frac{1}{2}\sqrt{2}$

- Regel
  1. $g^a \cdot g^b = g^{a+b}$
  2. $\frac{g^a}{g^b} = g^{a-b}$
  3. $(g^a)^b = g^{ab}$
  4. $(p \cdot q)^a = p^a \cdot q^a$
  5. $\left(\frac{p}{q}\right)^a = \frac{p^a}{q^a}$

Algemeen
$\frac{1}{g^n} = g^{-n}$

Algemeen
$\sqrt[n]{g} = g^{\frac{1}{n}}$

- $g^0 = 1$

- Vorm $A \cdot B = 0 \Rightarrow A = 0 \vee B = 0$

- Vorm $A \cdot B = C \cdot B \Rightarrow B = 0 \vee A = C$
  - Voorbeeld: $6(4x - 5) = (2x - 3)(4x - 5)$
    $2x - 3 = 6 \vee 4x - 5 = 0$
    $2x = 9 \quad \vee 4x = 5$
    $x = 4\frac{1}{2} \quad \vee x = 1\frac{1}{4}$

- Vorm $A^2 = B^2 \Rightarrow A = B \vee A = -B$
  - Voorbeeld: $(2x - 7)^2 = 9$
    $2x - 7 = 3 \vee 2x - 7 = -3$
    $2x = 10 \quad \vee 2x = 4$
    $x = 5 \quad \vee x = 2$

- Hogeregraadsvergelijkingen
  - Factor buiten haakjes halen
    - Voorbeeld: $x^3 - 7x^2 - 30x = 0$
      $x(x^2 - 7x - 30) = 0$
      $x = 0 \lor (x - 10)(x + 3) = 0$
      $x = 0 \lor\ x = 10 \lor x = -3$
  - Substitutie
    - Voorbeeld: $x^6 + 9x^3 - 10 = 0$
      Stel $x^3 = p$
      $p^2 + 9p - 10 = 0$
      $(p + 10)(p - 1) = 0$
      $p = -10 \lor\ p = 1$
      $x^3 = -10 \lor x^3 = 1$
      $x = \sqrt[3]{-10} \lor x = 1$

- Wortelfuncties
  - Voorbeeld: $f(x) = \sqrt{x}$
    - Domein: $D_f = [0, \rightarrow\rangle$
    - Bereik: $B_f = [0, \rightarrow\rangle$

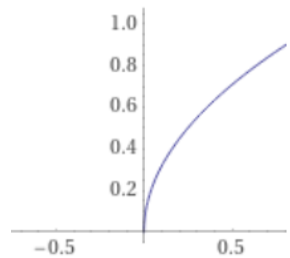
- Wortelvergelijkingen
  1. Wortel isoleren
  2. Links en rechts kwadrateren
  3. Antwoord controleren
     - $\sqrt{A} = B$ bestaat alleen voor $B \geq 0$

f(x) = sqrt(x)

Domain $x \in \mathcal{R} : x \geq 0$

Range $f \in \mathcal{R} : f \geq 0$

injective (one-to-one), that is, f is mapped to at most one x.

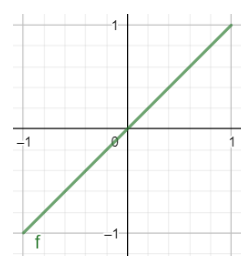


f(x) = x

Domain = R

Range = R

Bijective, odd parity

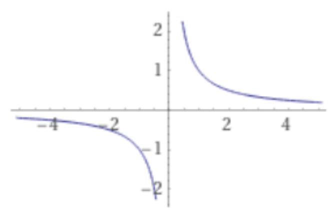


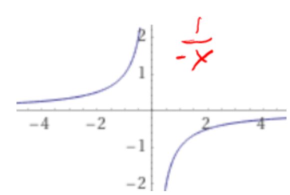
f(x) = 1/x

Domain $x \in \mathcal{R} :\ x \neq 0$

Range $f \in \mathcal{R} : f \neq 0$

Injective, odd parity, that is f(-x) = -f(x)

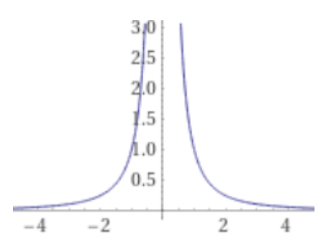




f(x) = 1/x^2 , even parity

Domain $x \in \mathcal{R} :\ x \neq 0$

Range $f \in \mathcal{R} : f > 0$



- Gebroken vergelijkingen
  - Kruiselings vermenigvuldigen: $\frac{a}{b} = \frac{c}{d} \;\Rightarrow\; ad = bc$
    - Controleren: $b \neq 0 \wedge d \neq 0$

- Functie van de vorm $f(x) = ax^2 + bx + c$
  - Kwadraat afsplitsen
  - Voorbeeld 1: $f(x) = x^2 - 6x + 2$
    - $f(x) = x^2 - 6x + 2$
      $= (x-3)^2 - 9 + 2$
      $= (x-3)^2 - 7$

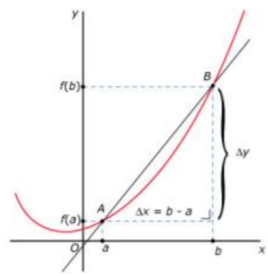
merkwaardige producten
$(x-p)^2 = x^2 - 2px + p^2$
$(x+p)^2 = x^2 + 2px + p^2$

- Absolute waarde
  - = Modulus
  - = Afstand op de getallenlijn tot 0
  - $|x| = \begin{cases} x & \text{als } x \geq 0 \\ -x & \text{als } x < 0 \end{cases}$

(x-3)(x+3) = x^2 – 3^2

- Aanpak
  - $|A| = c \quad \Rightarrow \quad A = c \vee A = -c$ (mits $c \geq 0$)
  - $|A| = |B| \;\Rightarrow\; A = B \vee A = -B$

- Helling van een grafiek
  - Lineaire grafiek: richtingscoëfficiënt
  - Niet-lineaire grafiek: helling verschilt van punt tot punt

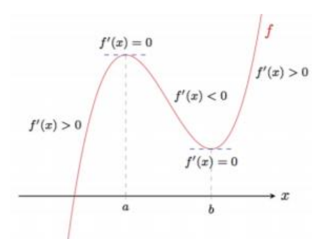


- Differentiequotiënt van f op $[a, b]$
  - $= \frac{\Delta f}{\Delta x} = \frac{f(b)-f(a)}{b-a}$
  - = *gemiddelde* helling van de grafiek van f op het interval $[a, b]$
  - = richtingscoëfficiënt van lijn AB

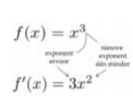
- Differentiaalquotiënt voor een willekeurige waarde van x berekenen
  - Voorbeeld: $f(x) = 8x - x^2$
  - $\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

- Differentiëren
  - Het berekenen van de afgeleide

- Afgeleide en helling
  - Als $f'(x) > 0$ dan is de grafiek van $f(x)$ stijgend
  - Als $f'(x) < 0$ dan is de grafiek van $f(x)$ dalend
  - Als $f'(x) = 0$ dan is de grafiek van $f(x)$ 'stationair'
    - Komt aan bod bij 'Extreme waarden'



- Standaardafgeleiden
  - Afgeleide van lineaire functies
    - $f(x) = ax + b \implies f'(x) = a$
  - Afgeleide van machtsfuncties (machtregel)
    - $f(x) = x^n \implies f'(x) = n \cdot x^{n-1}$



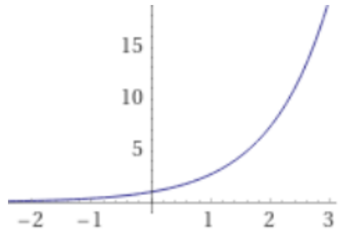
functie	afgeleide
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = x^3$	$f'(x) = 3x^2$
$f(x) = x^4$	$f'(x) = 4x^3$
$f(x) = x^5$	$f'(x) = 5x^4$
$f(x) = x^6$	$f'(x) = 6x^5$

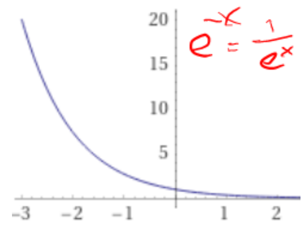
f(x) = e^x

Domain $\mathcal{R}$

Range $f \in \mathcal{R} : f > 0$

Injective





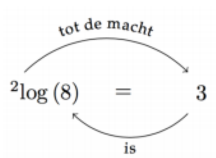
	Lineaire groei	Exponentiële groei
Formule	$N = at + b$ a = richtingscoëfficiënt b = beginwaarde	$N = b \cdot g^t$ g = groeifactor b = beginwaarde
Kenmerk	Constante toe- of afname a	Constante vermenigvuldigingsfactor g

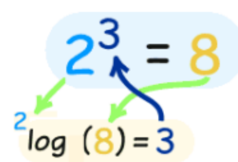
als g>1

- Rekenregels
  1. ${}^g\log(a) + {}^g\log(b) = {}^g\log(ab)$
  2. ${}^g\log(a) - {}^g\log(b) = {}^g\log\left(\frac{a}{b}\right)$
  3. $n \cdot {}^g\log(a) = {}^g\log(a^n)$
  4. ${}^g\log(a) = \frac{{}^p\log(a)}{{}^p\log(g)}$

Bewijs van rekenregel 1:

$$a \cdot b = ab$$
$$g^{{}^g\log(a)} \cdot g^{{}^g\log(b)} = g^{{}^g\log(ab)}$$
$$g^{{}^g\log(a) + {}^g\log(b)} = g^{{}^g\log(ab)}$$
$${}^g\log(a) + {}^g\log(b) = {}^g\log(ab)$$





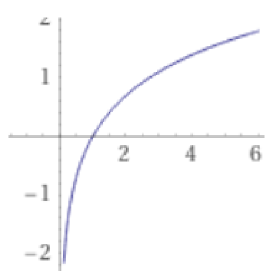
- Uit $g^a = b \Leftrightarrow a = {}^g\log(b)$ volgt nog:
  - ${}^g\log(g^a) = a$
  - $g^{{}^g\log(a)} = a$

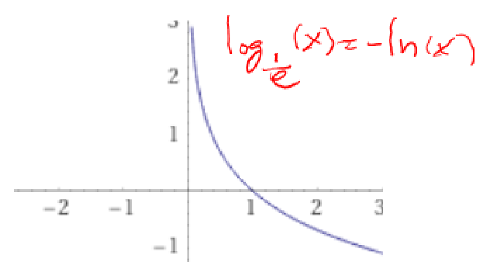
- Logaritmische vergelijkingen
  - Controleer je oplossingen!
    - ${}^g\log(a)$ bestaat alleen voor $a > 0$

f(x) = ln(x)

Domain $x \in \mathcal{R} : x > 0$

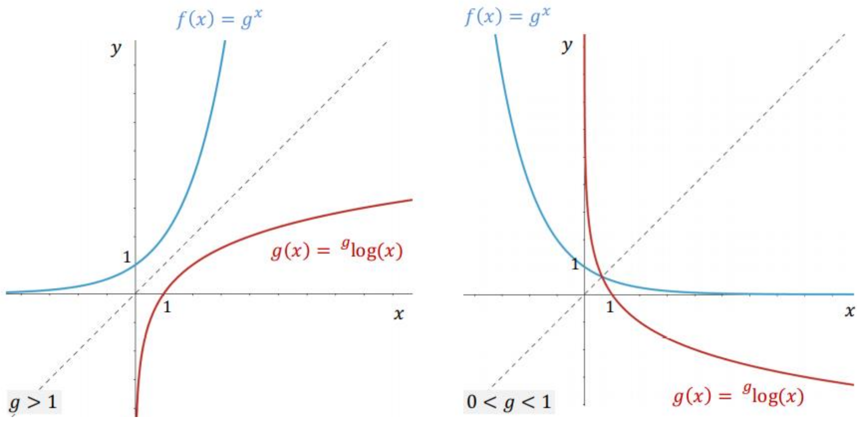
Range $\mathcal{R}$





Bijective, that is, it is injective (f is only mapped to one x), and surjective (all real numbers are represented in f)

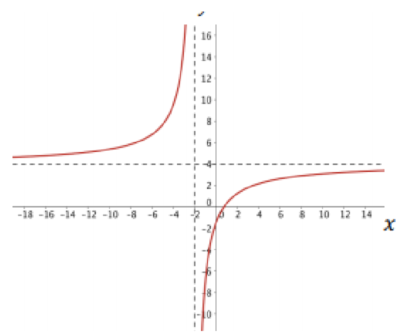
- Grafieken



$x \uparrow -2$ = x is smaller than 2 and x is increasing to approach 2 = x -> -2⁻

$x \downarrow -2$ = x is bigger than 2 and x is decreasing to approach 2 = x -> -2⁺

- **Gebroken functies**
  - Voorbeeld $f(x) = \dfrac{4x-3}{x+2}$
  - Verticale asymptoot
    - Voor $x = -2$ bestaat f niet
    - $\lim_{x \uparrow -2} f(x) = \infty$ en $\lim_{x \downarrow -2} f(x) = -\infty$



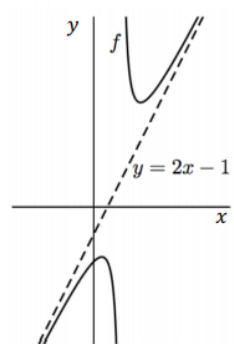
  - Horizontale asymptoot
    - Voor heel grote positieve (en heel grote negatieve) x geldt: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{4x-3}{x+2} \approx \dfrac{4x}{x} = 4$
    - De grafiek nadert daarom $y = 4$

- **Vergelijkingen van asymptoten bepalen**
  - Verticale asymptoot
    - Waarde(n) van x waarvoor de noemer 0 wordt (en de teller niet)
  - Horizontale asymptoot
    - Uitkomst van de functie voor grote positieve en/of grote negatieve waarden

Limiet		Voorwaarde	Voorbeeld
$\lim_{x\to-\infty}\left(\frac{a}{x^n}\right)=0$	$\lim_{x\to\infty}\left(\frac{a}{x^n}\right)=0$	$n>0$	$\lim_{x\to\infty}\left(\frac{5}{x^2}\right)=0$
$\lim_{x\to\infty}\left(\frac{a}{g^x}\right)=0$	$\lim_{x\to-\infty}(a\cdot g^x)=0$	$g>1$	$\lim_{x\to\infty}\left(\frac{1}{3^x}\right)=0$
$\lim_{x\to-\infty}\left(\frac{a}{g^x}\right)=0$	$\lim_{x\to\infty}(a\cdot g^x)=0$	$0<g<1$	$\lim_{x\to\infty}(2\cdot 0{,}8^x)=0$

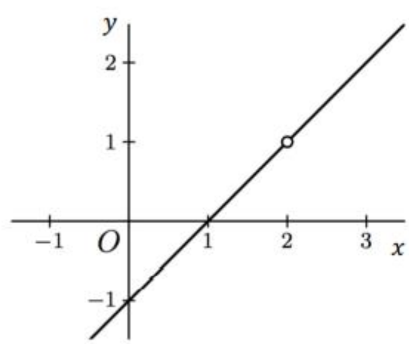
- Scheve asymptoot
  - Als de teller één graad hoger is dan de noemer
  - Voorbeeld 1: $f(x) = \dfrac{2x^2 - 3x + 2}{x - 1}$
  - Herleiden m.b.v. een staartdeling: $x - 1 \,/\, 2x^2 - 3x + 2 \,\backslash\, 2x - 1$

$$\begin{array}{r} 2x^2 - 3x + 2 \\ \underline{2x^2 - 2x\phantom{{}+2}} \\ -x + 2 \\ \underline{-x + 1} \\ 1 \end{array}$$



  - Dus $f(x) = 2x - 1 + \dfrac{1}{x-1}$
  - $\lim\limits_{|x|\to\infty} \dfrac{1}{x-1} = \lim\limits_{|x|\to\infty} \dfrac{\frac{1}{x}}{1 - \frac{1}{x}} = 0$
  - Als $|x| \to \infty$ nadert de grafiek van f de lijn $y = 2x - 1$.

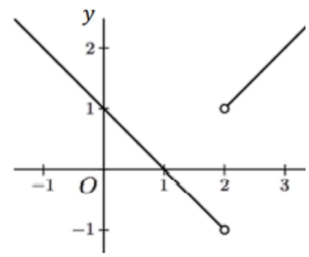
- Perforatie
  - Als zowel de teller als noemer voor $x = a$ naar 0 gaan, kan de grafiek daar een perforatie hebben
  - Voorbeeld: $h(x) = \dfrac{x^2 - 3x + 2}{x - 2} = \dfrac{(x-2)(x-1)}{x-2} = x - 1$ mits $x \neq 2$
    - $\lim\limits_{x\to 2} h(x) = 2 - 1 = 1$
    - Dus de grafiek heeft perforatie in het punt (2, 1)



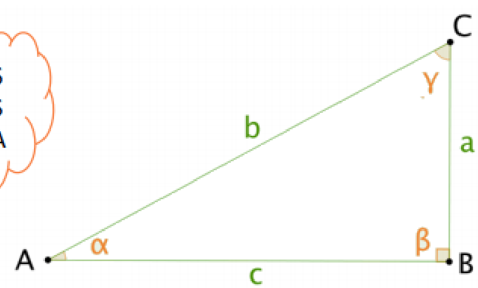
  - Voorwaarde: $\lim\limits_{x\to a} f(x)$ moet wél bestaan
    - Voorbeeld: $f(x) = \frac{x}{x^2}$
    - $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x} = \infty$
    - Dus de grafiek van f heeft geen perforatie bij $x = 0$

1 Theorem $\lim\limits_{x\to a} f(x) = L$ if and only if $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$

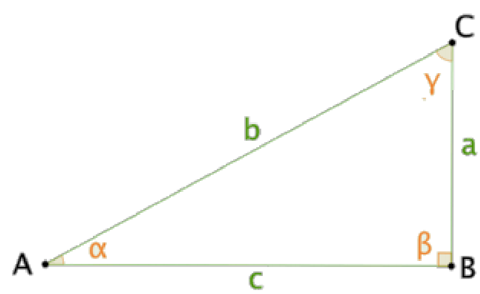
- Perforatie
  - Als zowel de teller als noemer voor $x = a$ naar 0 gaan, kan de grafiek daar een perforatie hebben
  - Voorbeeld: $j(x) = \dfrac{x^2 - 3x + 2}{|x-2|}$
    - Als $x \geq 2$ dan $j(x) = \dfrac{x^2 - 3x + 2}{x - 2} = \dfrac{(x-2)(x-1)}{x-2} = x - 1$ mits $x \neq 2$
    - Als $x < 2$ dan $j(x) = \dfrac{x^2 - 3x + 2}{-(x - 2)} = \dfrac{(x-2)(x-1)}{-(x-2)} = -x + 1$ mits $x \neq 2$
    - Sprong bij $x = 2$
      - $\lim_{x\downarrow 2} j(x) = 2 - 1 = 1$
      - $\lim_{x\uparrow 2} j(x) = -2 + 1 = -1$



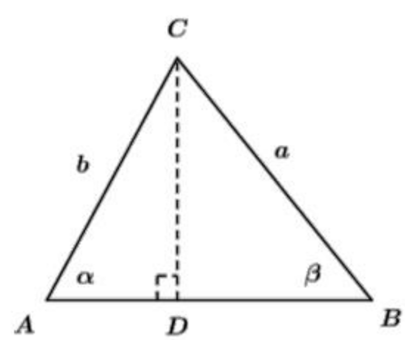
- In een rechthoekige driehoek ABC geldt:
  - $\sin(\alpha) = \frac{a}{b}$ $\left(\frac{\text{overstaande rechthoekszijde}}{\text{schuine zijde}}\right)$
  - $\cos(\alpha) = \frac{c}{b}$ $\left(\frac{\text{aanliggende rechthoekszijde}}{\text{schuine zijde}}\right)$
  - $\tan(\alpha) = \frac{a}{c}$ $\left(\frac{\text{overstaande rechthoekszijde}}{\text{aanliggende rechthoekszijde}}\right)$



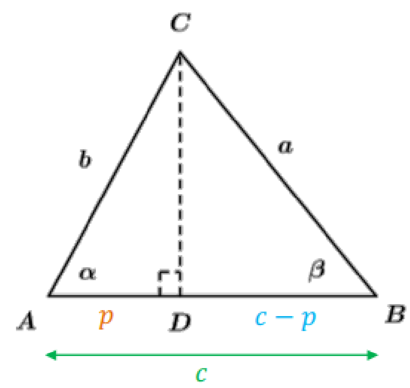
- In een rechthoekige driehoek ABC geldt bovendien de stelling van Pythagoras: $a^2 + b^2 = c^2$
  - Met c de schuine zijde (*hypothenusa*)



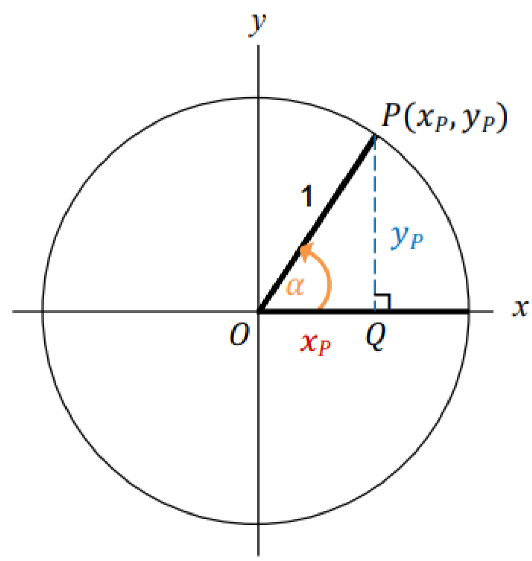
- Sinusregel: $\dfrac{a}{\sin(\alpha)} = \dfrac{b}{\sin(\beta)} = \dfrac{c}{\sin(\gamma)}$
  - Zinvol als een hoek en *tegenoverliggende* zijde bekend zijn



- Cosinusregel: $a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha)$
  - $b^2 = a^2 + c^2 - 2ac \cdot \cos(\beta)$
  - $c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$
  - Zinvol als 3 zijden bekend zijn
  - Zinvol als 2 zijden en een *ingesloten* hoek bekend zijn



- Cirkel met $M(0,0)$ en $r = 1$
  - Positieve draaiing linksom, t.o.v. positieve x-as
  - $\cos(\alpha) = \frac{OQ}{OP} = \frac{x_P}{1} = x_P$
  - $\sin(\alpha) = \frac{PQ}{OP} = \frac{y_P}{1} = y_P$
  - $\tan(\alpha) = \frac{PQ}{OQ} = \frac{y_P}{x_P} = \frac{\sin(\alpha)}{\cos(\alpha)}$



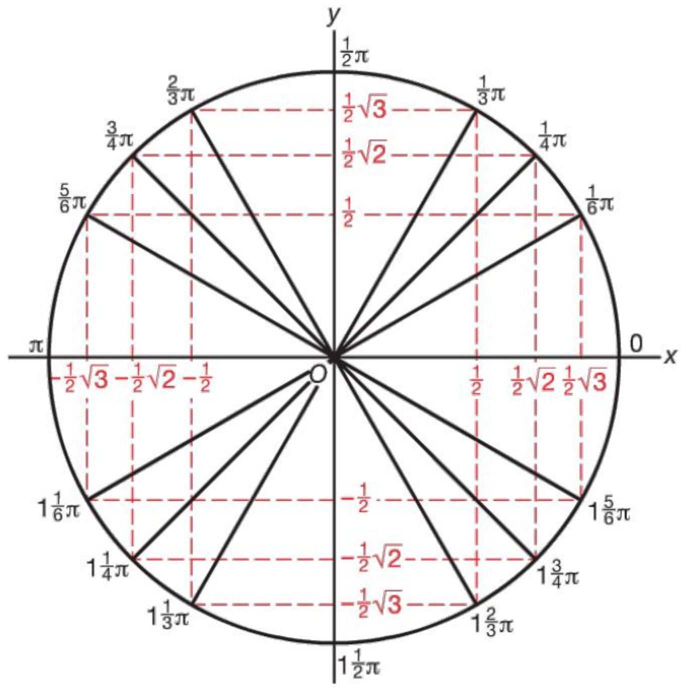
- Hoekeenheid radiaal
  - 1 radiaal is de grootte van de middelpuntshoek waarbij de lengte van de cirkelboog gelijk is aan de straal van de cirkel

- Omrekenen
  - Middelpuntshoek 360° geeft in de eenheidscirkel cirkelboog $2\pi \cdot 1$
  - $360° = 2\pi$ rad

°	360°	180°	90°	60°	45°	30°	57,3°
rad	2π	π	$\frac{1}{2}\pi$	$\frac{1}{3}\pi$	$\frac{1}{4}\pi$	$\frac{1}{6}\pi$	1

  - Geen eenheid vermeld ⇒ radiaal
    - Vermeld dus altijd gradenbolletjes als de hoek in graden is!

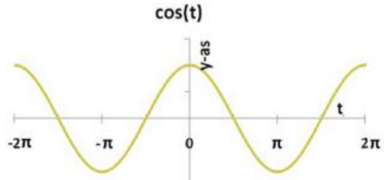
- Rekenregels uit de eenheidscirkel
  1. $\sin^2(\alpha) + \cos^2(\alpha) = 1$ $\quad \cot x = \dfrac{1}{\tan x}$
  2. $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ $\quad \cot x = \dfrac{\cos x}{\sin x}$
  3. $-\sin(\alpha) = \sin(\alpha - \pi) = \sin(-\alpha)$
  4. $-\cos(\alpha) = \cos(\alpha - \pi) = \cos(\pi - \alpha)$
  5. $\sin(\alpha) = \cos\left(\frac{1}{2}\pi - \alpha\right)$
  6. $\cos(\alpha) = \sin\left(\frac{1}{2}\pi - \alpha\right)$
     - 5. en 6.: complementformules



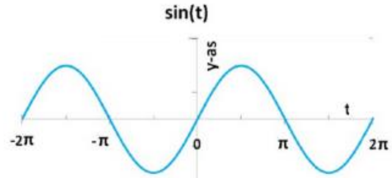
- Exacte waarden in de eenheidscirkel
  - $\begin{cases} x = \cos(\alpha) \\ y = \sin(\alpha) \end{cases}$

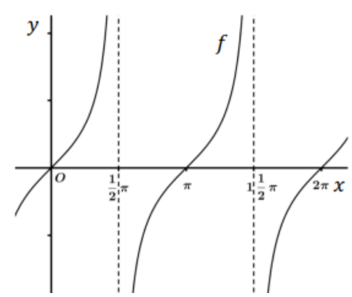
α in °	0°	30°	45°	60°	90°
α in rad	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
$\cos(\alpha)$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0

cos(t)



sin(t)





- Evenwichtsstand is 0
- Periode is 2π
  - $f(x) = f(x + k \cdot 2\pi)$
- Amplitude is 1
- Startpunt
  - Sinus → stijgend door de evenwichtsstand
  - Cosinus → maximum

- Verticale asymptoot als $\cos(x) = 0$
  - $x = \frac{1}{2}\pi, t = 1\frac{1}{2}\pi, \ldots$
- Periode: π

$\sin(A) = \sin(B) \Rightarrow A = B + k \cdot 2\pi \vee A = \pi - B + k \cdot 2\pi$

$\cos(A) = \cos(B) \Rightarrow A = B + k \cdot 2\pi \vee A = -B + k \cdot 2\pi$

- $\tan(A) = \tan(B)$ geeft $A = B + k \cdot \pi$

$\sin(t+u) = \sin(t)\cos(u) + \cos(t)\sin(u)$

$\sin(t-u) = \sin(t)\cos(u) - \cos(t)\sin(u)$

$\cos(t+u) = \cos(t)\cos(u) - \sin(t)\sin(u)$

$\cos(t-u) = \cos(t)\cos(u) + \sin(t)\sin(u)$

$\sin(2t) = 2\sin(t)\cos(t)$

$\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 = 1 - 2\sin^2(t)$

Week 1 Functions, Limits and Continuity

Chapter 1. Functions – Ignore this intro, read Chapter 12 of Book of Proof instead

1. verbally (by a description in words)
2. numerically (by a table of values)
3. visually (by a graph)
4. algebraically (by an explicit formula)

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years. For instance,

$$P(1950) \approx 2{,}560{,}000{,}000$$

But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

A function describes a rule whereby, given a number (r, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number.

A function f is a rule that assigns to each element x in a set D exactly one element, called f(x), in a set E.

The set D is called the domain of the function. The range of f is the set of all possible values of f(x).

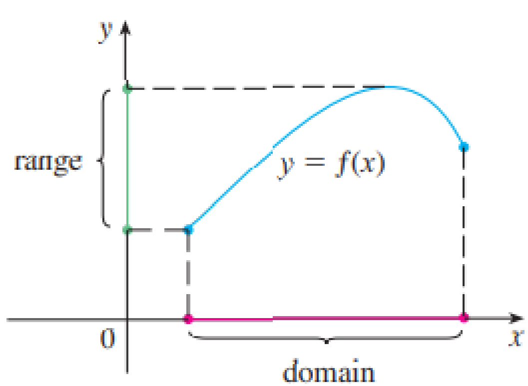
A symbol that represents a number in the domain is called an independent variable

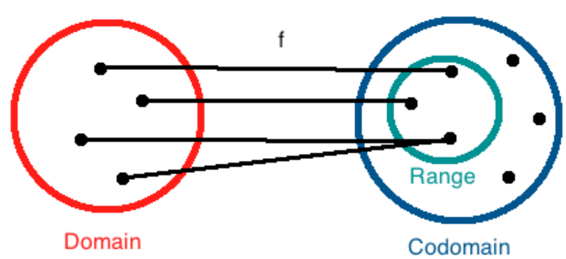
A symbol that represents a number in the range is called a dependent variable.

We can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs:

$\{(x, f(x)) \mid x \in D\}$





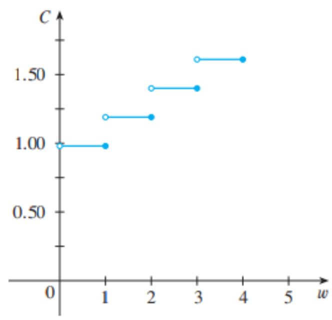
piecewise defined functions:

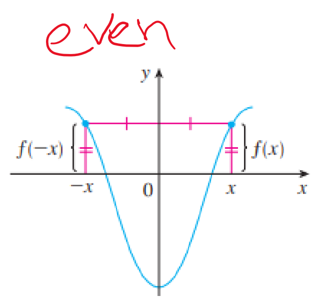
$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

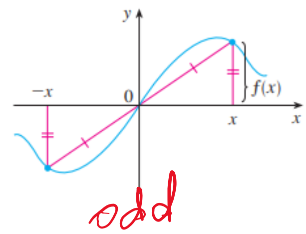
absolute value:

$$|a| = a \quad \text{if } a \geqslant 0$$
$$|a| = -a \quad \text{if } a < 0$$

step functions:







Odd symetry/parity: If a function satisfies f(-x) = -f(x) for every number x in its domain. Such as f(x) = x^3

Even symetry/parity: If a function f satisfies f(-x) = f(x) for every number x in its domain. Such as f(x)=x^2

Vertical test: check if it is a function

Horizontal test: check if it is injective (1-to-1)

bijectiveness is up to how the co-domain was defined

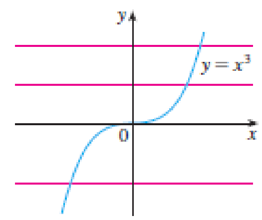


FIGURE 3
$f(x) = x^3$ is one-to-one.

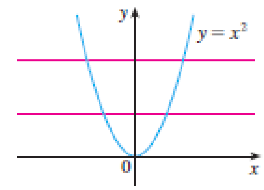
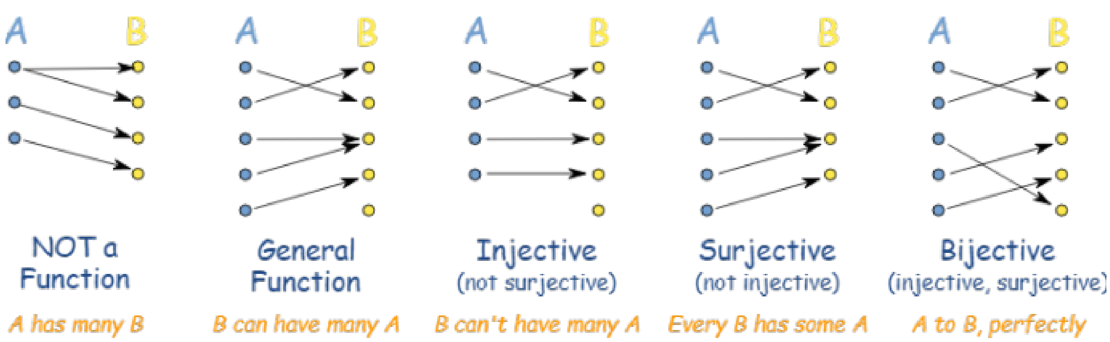
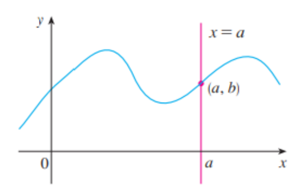
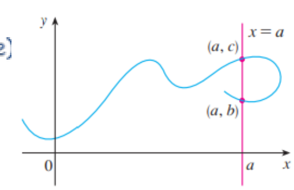


FIGURE 4
$g(x) = x^2$ is not one-to-one.





(a) This curve represents a function.



(b) This curve doesn't represent a function.

Chapter 1.5 Inverse Functions

The inverse function of f , denoted by f $^{-1}$, and read " f inverse." Not all functions possess inverses. The function has to be injective (one-to-one) that is, each input x has it's own unique output y. Otherwise, if we had two x's having the same y, if we "swap" x for y, we are left with two y's having one x, in other words, an input providing two outcomes. That is not a valid function.

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

4

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

We must respect the domains.

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

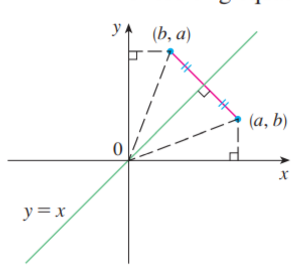


FIGURE 8

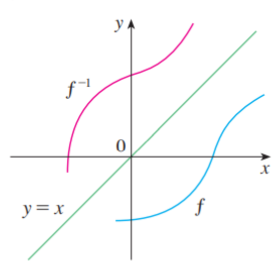
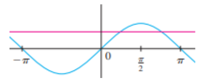
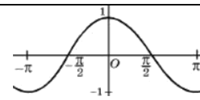
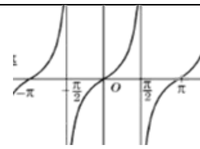
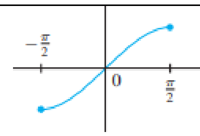
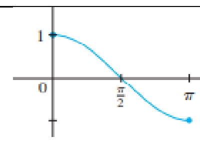
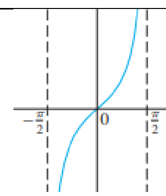
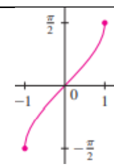
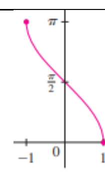
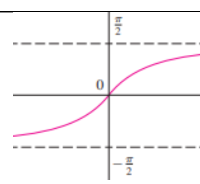
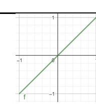


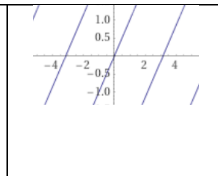
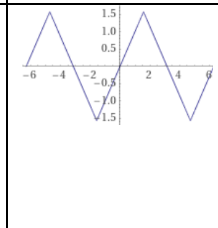
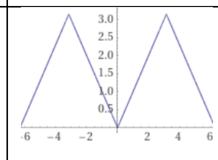
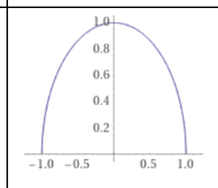
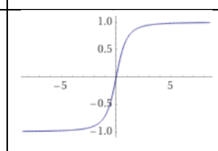
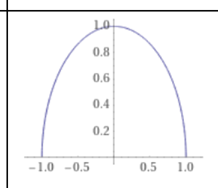
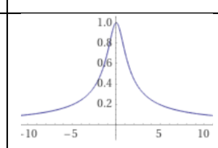
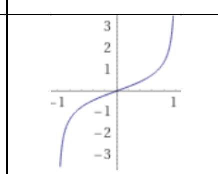
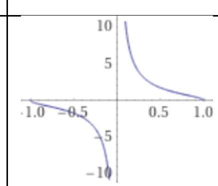
FIGURE 9

Therefore, as illustrated by Figure 9:

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

*relation with the co-domain and range, for inverse functions assumed co-domain = R

function	domain	range/(co-d)	graph	parity	relation
sin(angle)=ratio = $\sin(\alpha + k \cdot 2\pi)$	R *period, can change angle	[-1,1] (R)		odd	general functn
cos(α)=ratio = $\cos(\alpha + k \cdot 2\pi)$	R *period, can change α	[-1,1] (R)		even	general functn
tan(α)=ratio = $\tan(\alpha + k \cdot \pi)$	R except $\frac{1}{2}\pi + k * \pi$ *can change α	[-inf,inf] (R)		odd	srjctve
one-to-one sin x	*[$-\frac{1}{2}\pi, \frac{1}{2}\pi$]*	*[-1,1]* *(R)*		*odd*	*injctve*
one-to-one cos x	*[0, π]*	*[-1,1]* *(R)*		*none*	*injctve*
one-to-one tan x	*($-\frac{1}{2}\pi, \frac{1}{2}\pi$)*	*(-inf,inf) so R* *(R)*		*odd*	*bijctve*
arcsin(ratio)=angle	[-1,1] *No period can't change ratio	[$-\frac{1}{2}\pi, \frac{1}{2}\pi$] (R)		odd	injctve
arccos(r)=angle	[-1,1] *No period can't change r	[0, π] (R)		none	injctve
arctan(r)=angle	R *No period can't change r	($-\frac{1}{2}\pi, \frac{1}{2}\pi$) (R)		odd	injctve
sin(arcsin(x))=x cos(arccos(x))=x	[-1,1] *No period can't change x	[-1,1] (R)		odd	injctve

$arctan(tan(\alpha)) =$ $-\frac{1}{2}\pi < \alpha + \pi \cdot k < \frac{1}{2}\pi$ *can change α	R except $\frac{1}{2}\pi + k * \pi$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ (R)		odd	general functn
$arcsin(sin(\alpha)) =$ $-\frac{1}{2}\pi \le \alpha + k2\pi \le \frac{1}{2}\pi$ sin(-α) = $-$ sin(α) sin(α) = $-$ sin($\alpha \pm \pi$) *sin(α) = sin($\pi - \alpha$)*	R *period, can change angle	$[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ (R)		odd	general functn
$arccos(cos(\alpha)) =$ $0 \le \alpha + k \cdot 2\pi \le \pi$ cos(α)=cos(-α) cos(α)= $-$ cos($\alpha \pm \pi$)	R *period, can change angle	[0, π] (R)		even	general functn
tan(arctan(r))=angle	R *No period can't change r	R (R)	f(x)=x	odd	bijctve
$\sin(\arccos(r)) =$ $= \sqrt{1 - r^2}$	[-1,1] *No period can't change r	[0,1] (R)		even	general functn
$sin(arctan(r))$ $= \frac{r}{\sqrt{1+r^2}}$	R *No period can't change r	[-1,1] (R)		odd	injctve
$\cos(arcsin(r))$ $= \sqrt{1 - r^2}$	[-1,1] *No period can't change r	[0,1] (R)		even	general functn
$cos(arctan(r))$ $= \frac{1}{\sqrt{1+r^2}}$	R *No period can't change r	[0,1] (R)		even	general functn
$tan(arcsin(r))$ $= \frac{r}{\sqrt{1-r^2}}$	(-1,1) *No period can't change r	R (R)		odd	bijctve
$tan(arccos(r))$ $= \frac{\sqrt{1-r^2}}{r}$	[-1,0) ∧ (0,1] * No period can't change r	R (R)		odd	srjctve ? lim ± inf = 0

Lecture. Limits

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in the exercises for Section 2.4.

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows. (For square roots the proof is outlined in Exercise 2.4.37.)

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem

SQUEEZE THEOREM

If

$$h(x) \leq f(x) \leq g(x)$$

And

$$\lim_{x \to a} h(x) = L$$

And

$$\lim_{x \to a} g(x) = L$$

Then

$$\lim_{x \to a} f(x) = L$$

We know that

$$-1 \leq \sin \frac{\pi}{x} \leq 1$$

Multiply throughout by $\sqrt{x^3 + x^2}$

$$-\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \sin \frac{\pi}{x} \leq \sqrt{x^3 + x^2}$$

Since

$$\lim_{x \to 0} -\sqrt{x^3 + x^2} = 0$$

And

$$\lim_{x \to 0} \sqrt{x^3 + x^2} = 0$$

By **Squeeze Theorem**, We have

$$\lim_{x \to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Lecture. Continuity

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

7 Theorem The following types of functions are continuous at every number in their domains:

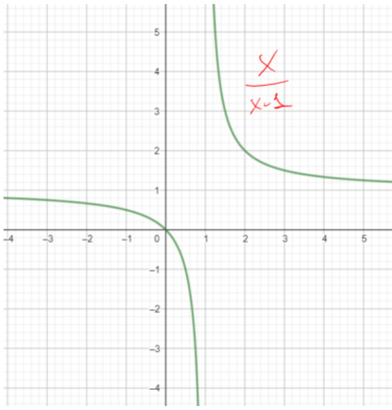
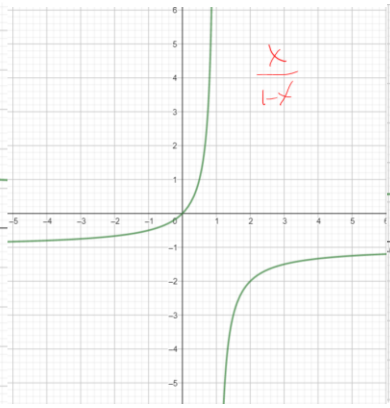
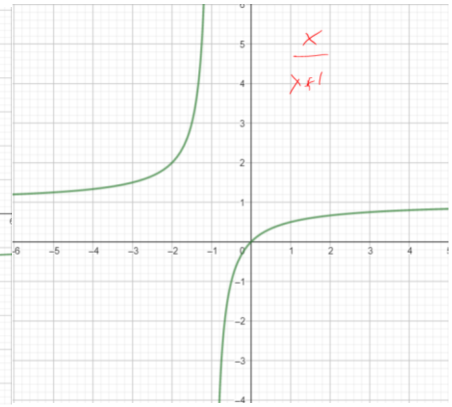
- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

8 Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

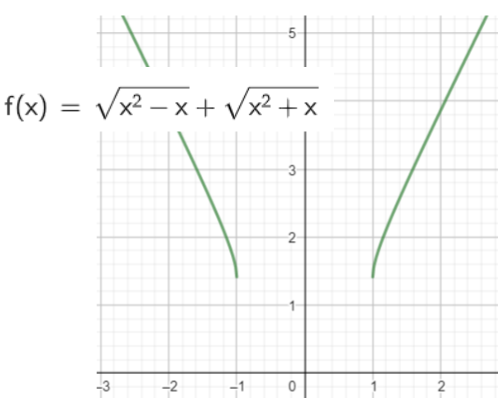




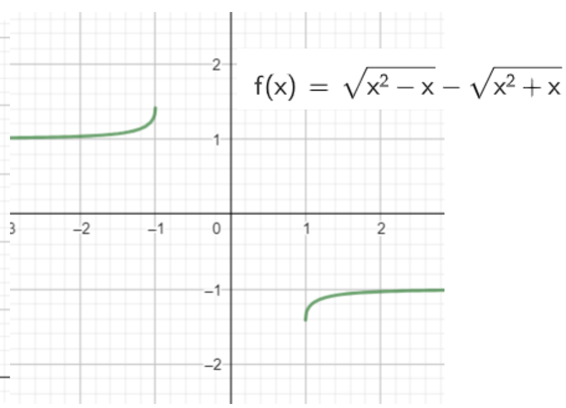
Evaluate: $\lim\limits_{x\to-\infty} \sqrt{x^2+7\cdot x+3}-\sqrt{x^2-9\cdot x+0}$

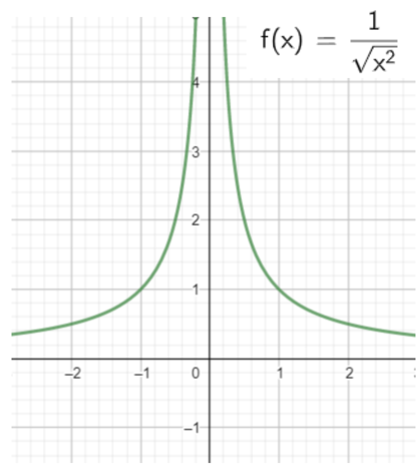
Your answer: 8

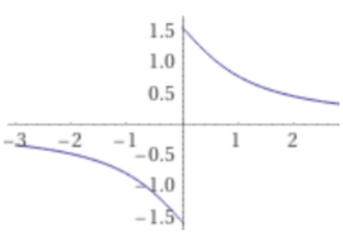
Your answer is incorrect. The correct answer is -8.

Multiply both the numerator and the denominator by $\sqrt{x^2+7\cdot x+3}+\sqrt{x^2-9\cdot x+0}$. Then we obtain: $\lim\limits_{x\to-\infty} \sqrt{x^2+7\cdot x+3}-\sqrt{x^2-9\cdot x+0}=\lim\limits_{x\to-\infty} \dfrac{16x+3}{\sqrt{x^2+7\cdot x+3}+\sqrt{x^2-9\cdot x+0}}=-\dfrac{16}{2}$. In the last step we divide both the numerator and the denominator by $-x=\sqrt{x^2}$ (since $x<0$).



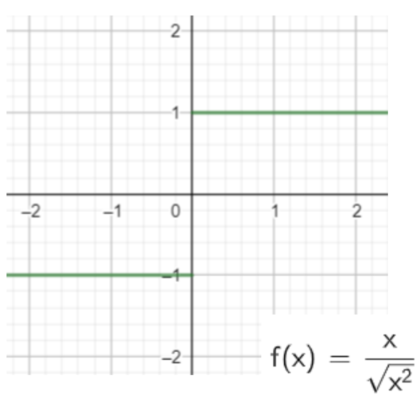


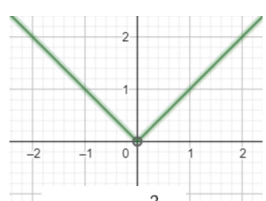




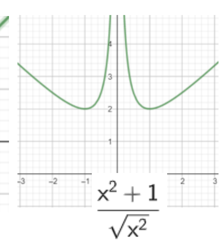
$f(x)=\arctan(\frac{1}{x})$

no inf = no vert asymptote.



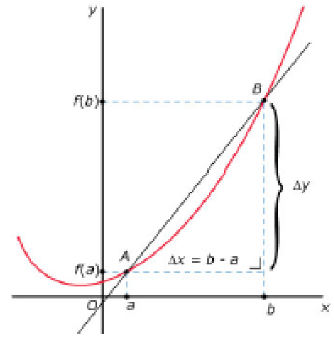


$f(x)=\frac{x^2}{\sqrt{x^2}}$



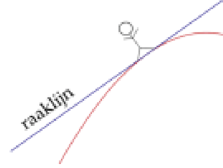
Wiskunde B refresher 2. Afgeleide

- Helling van een grafiek
  - Lineaire grafiek: richtingscoëfficiënt
  - Niet-lineaire grafiek: helling verschilt van punt tot punt



- Differentiequotiënt van f op $[a, b]$
  - $= \frac{\Delta f}{\Delta x} = \frac{f(b)-f(a)}{b-a}$
  - = *gemiddelde* helling van de grafiek van f op het interval $[a, b]$
  - = richtingscoëfficiënt van lijn AB

- Helling van een grafiek in één bepaald punt
  - Richtingscoëfficiënt van de raaklijn



- Differentiaalquotiënt voor een willekeurige waarde van x berekenen
  - Voorbeeld: $f(x) = 8x - x^2$
  - $$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

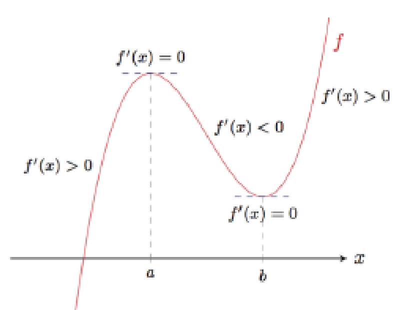
$$= \lim_{h \to 0} \frac{8(x+h) - (x+h)^2 - (8x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{8x + 8h - x^2 - 2xh - h^2 - 8x + x^2}{h}$$

$$= \lim_{h \to 0} \frac{8h - 2xh - h^2}{h} = \lim_{h \to 0} (8 - 2x - h) = 8 - 2x = f'(x)$$

- Afgeleide van f
  - Functievoorschrift voor de helling van f (check: $f'(3) = 8 - 2 \cdot 3 = 2$)
  - Notatie: $f'(x)$ of $\frac{\mathrm{d}f}{\mathrm{d}x}$

- Differentiëren
  - Het berekenen van de afgeleide

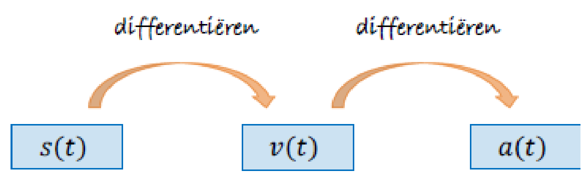


- Afgeleide en helling
  - Als $f'(x) > 0$ dan is de grafiek van $f(x)$ stijgend
  - Als $f'(x) < 0$ dan is de grafiek van $f(x)$ dalend
  - Als $f'(x) = 0$ dan is de grafiek van $f(x)$ 'stationair'
    - Komt aan bod bij 'Extreme waarden'

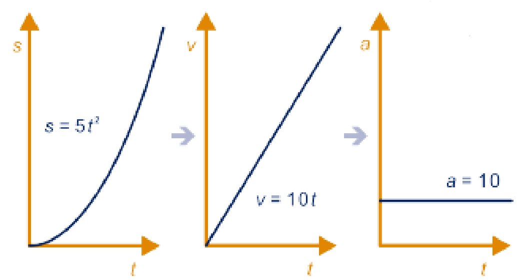
- Standaardafgeleiden
  - Afgeleide van lineaire functies
    - $f(x) = ax + b \implies f'(x) = a$
  - Afgeleide van machtsfuncties (machtregel)
    - $f(x) = x^n \implies f'(x) = n \cdot x^{n-1}$

- Differentiëren van machtsfuncties met negatieve en gebroken exponenten
  1. Herleid (een deel van) de functie tot de vorm $f(x) = ax^n$
  2. Differentieer m.b.v. de machtregel
  3. Herleid afgeleide tot een vorm zonder negatieve en gebroken exponent

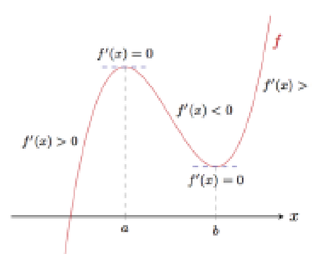
- Bewegingen
  - Afgelegde weg: $s(t)$
  - Snelheid: $v(t) = s'(t)$
  - Versnelling: $a(t) = v'(t)$

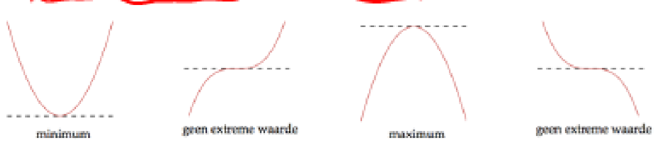


  - Voorbeeld
    - Eenparig versnelde beweging



- Functie f heeft een extreme waarde
  - Functie f heeft een minimum of maximum
  - Grafiek van f heeft een top
    - Dan geldt: $f'(x) = 0$
    - Let op: andersom is dit niet altijd zo!





Type functie	Functie	Afgeleide
Constante	$f(x) = a$	$f'(x) = 0$
Machtsfunctie	$f(x) = x^n$	$f'(x) = n \cdot x^{n-1}$
	$f(x) = \frac{1}{x}$	$f'(x) = -\frac{1}{x^2}$
Exponentiële functie	$f(x) = e^x$	$f'(x) = e^x$
	$f(x) = g^x$	$f'(x) = g^x \cdot \ln(g)$
Logaritmische functies	$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
	$f(x) = {}^g\log(x)$	$f'(x) = \frac{1}{x\ln(g)}$
Goniometrische functies	$f(x) = \sin(x)$	$f'(x) = \cos(x)$
	$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
	$f(x) = \tan(x)$	$f'(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$

- Differentieerregels

Regel	Functie	Afgeleide
Constante erbij optellen	$f(x) = g(x) + c$	$f'(x) = g'(x)$
Met een constante vermenigvuldigen	$f(x) = c \cdot g(x)$	$f'(x) = c \cdot g'(x)$
Somregel	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Productregel	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotiëntregel	$q(x) = \frac{t(x)}{n(x)}$	$q'(x) = \frac{n(x) \cdot t'(x) - t(x) \cdot n'(x)}{(n(x))^2}$
Kettingregel	$k(x) = f(u(x))$	$k'(x) = f'(u(x)) \cdot u'(x)$

Week 2. Differentiation

Chapter 3.5 Implicit differentiation

Implicit differentiation: differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

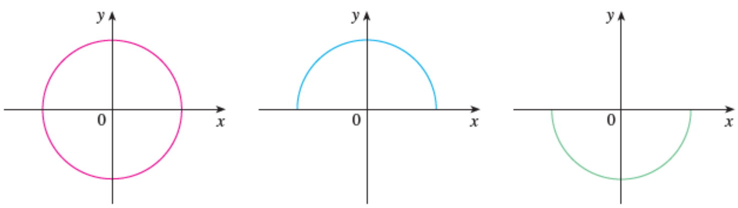
or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

1 $$x^2 + y^2 = 25$$

or

2 $$x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and $g(x) = -\sqrt{25 - x^2}$. The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$. (See Figure 1.)



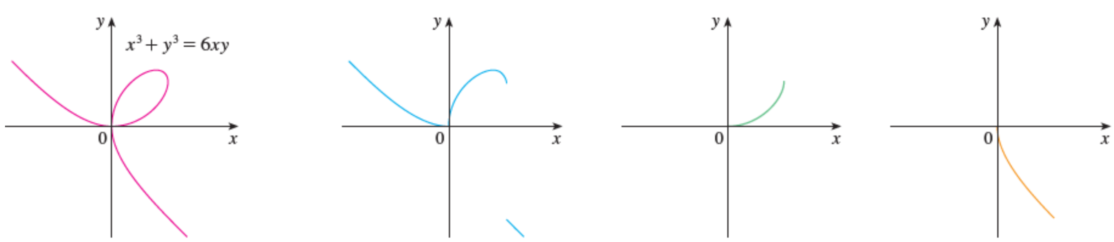
(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

$$x^2 + y^2 = 25$$
$$x^2 + \left(f(x)\right)^2 = 25$$
$$\left[x^2 + \left(f(x)\right)^2\right]' = [25]'$$
$$[x^2]' + \left[\left(f(x)\right)^2\right]' = 0$$
$$2x + 2f(x) \cdot f'(x) = 0$$
$$2f(x) \cdot f'(x) = -2x$$
$$f'(x) = \frac{-2x}{2f(x)}$$
$$f'(x) = -\frac{x}{f(x)} = -\frac{x}{y}$$

It's not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, (2) is the equation of a curve called the **folium of Descartes** shown in Figure 2 and it implicitly defines y as several functions of x. The graphs of three such functions are shown in Figure 3. When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + [f(x)]^3 = 6xf(x)$$

is true for all values of x in the domain of f.



Pay attention to the fact that y is in fact f(x) and thus the **chain rule holds**.

if f is any one-to-one differentiable function,it can be proved that its inverse function f^{-1} is also differentiable, except where its tangents are vertical.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

Determine the derivative of f^{-1} by implicitly differentiating the equation $f(y) = x$ with respect to x.

Your answer: $-\frac{1}{8\sqrt{y}}$

Your answer is incorrect. The correct answer is $-\frac{1}{2\cdot\sqrt{2\cdot x}}$.

In this case, $f(y) = x$ gives $2y^2 = x$, with $y \leq 0$ (the domain of function f, as given in the exercise).
Implicitly differentiating of $2y^2 = x$ with respect to x (in this case with $y = f^{-1}(x)$) gives: $2\cdot 2\cdot y\cdot y' = 1$. Hence, $y' = \frac{1}{2\cdot 2\cdot y}$.

Now, use $2y^2 = x$ to fill in $y = -\sqrt{\frac{x}{2}}$.

This yields $\left(f^{-1}\right)'(x) = -\frac{1}{2\cdot\sqrt{2\cdot x}}$.

Given the invertible function $f(x) = 2x^3$.

First show that this function is strictly increasing, and hence invertible.

Find the derivative of the inverse of this function at 16.

Hint: if $f(a) = b$ then $\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$.

Find $\left(f^{-1}\right)'(16)$. Give the exact value.

Your answer: 2

Your answer is incorrect. The correct answer is $\frac{1}{24}$.

$$\left(f^{-1}\right)'(16) = \frac{1}{f'(2)} \text{ because } f(2) = 16\,.$$

The formula for the tangent line at $(2,4)$ is $y - 4 = r(x - 2)$ with $r = \frac{dy}{dx}\Big|_{(2,4)}$

Chapter 3.10 Linear Approximations and Differentials

1 $$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

2 $$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

(Which is just the formula for the tangent line)

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (1) is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

$$|f(x) - L(x)| < accuracy \qquad OR \qquad f(x) - accuracy < L(x) < f(x) + accuracy$$

EXAMPLE 2 For what values of x is the linear approximation

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4}$$

accurate to within 0.5? What about accuracy to within 0.1?

SOLUTION Accuracy to within 0.5 means that the functions should differ by less than 0.5:

$$\left| \sqrt{x + 3} - \left(\frac{7}{4} + \frac{x}{4} \right) \right| < 0.5$$

Equivalently, we could write

$$\sqrt{x + 3} - 0.5 < \frac{7}{4} + \frac{x}{4} < \sqrt{x + 3} + 0.5$$

3 $$dy = f'(x)\,dx$$

So dy is a dependent variable

Approximate $\sqrt[3]{1001}$

for small values of Δx, we can say that

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} \approx f'(x)$$

$$f(x+\Delta x) \approx f(x) + \Delta x \cdot f'(x)$$

Let $f(x) = \sqrt[3]{x}$, $\Delta x = 1$, $x = 1000$

Then we have

$$f(1000+1) \approx f(1000) + (1) \cdot f'(1000)$$

$$\sqrt[3]{1001} \approx \sqrt[3]{1000} + f'(1000)$$

$$\sqrt[3]{1001} \approx 10 + f'(1000) \rightarrow \mathbf{(1)}$$

The only Unknown we have is $f'(1000)$

$$f(x) = \sqrt[3]{x}$$

$$f(x) = x^{1/3}$$

Differentiate

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f'(x) = \frac{x^{1/3}}{3x}$$

$$f'(1000) = \frac{(1000)^{1/3}}{3000}$$

$$f'(1000) = \frac{10}{3000} = \frac{1}{300}$$

Substitute the value of $f'(1000)$ in Eqn$\mathbf{(1)}$

$$\sqrt[3]{1001} \approx 10 + \frac{1}{300}$$

$$\sqrt[3]{1001} \approx \frac{3000+1}{300} = \frac{3001}{300}$$

33. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

$$V(x) = x^3$$

Where x is the length of edge of the cube

Differentiate
$$\frac{dV}{dx} = 3x^2$$

When Δx is small, like in this case, we can write

$$\frac{\Delta V}{\Delta x} \approx 3x^2$$

$$\Delta V \approx 3x^2 \cdot \Delta x$$

$\text{Maximum Error} = 3 \cdot 30^2 \cdot 0.1 = 270\ \text{cm}^3$

$$\text{Relative Error} = \frac{\text{Error}}{\text{Volume}} = \frac{270}{30^3} = 0.01$$

$$\text{Percentage Error} = \text{Relative Error} \times 100 = 0.01 \times 100 = 1\%$$

$\text{Maximum Error} = 270\ \text{cm}^3$

$\text{Relative Error} = 0.01$

$\text{Percentage Error} = 1\%$

Volume of a cylinder

A company constructs cylindrical steel beams with a lenght of 8 meter. The diameter of each beam is equal to 0.6 meter. How much would the volume of a beam approximately increase if, because of a construction error, the diameter would incease by 0.01 meter?

Let V be the volume of the beam and x the diameter.

$$V = 8\pi\left(\frac{x}{2}\right)^2 \qquad dV = 4\pi x dx$$

Using the formula for the differential we find that the volume increase is approximately

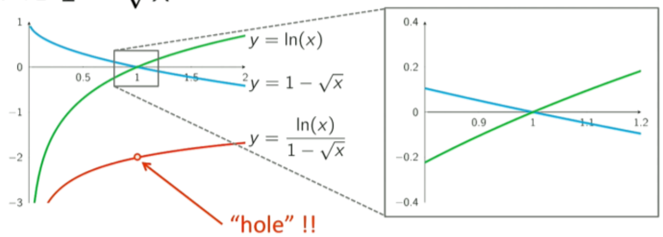
$$dV = 4\pi(0.6)(0.01)m^3 = 0.0754m^3$$

If we divide this error by the total volume V we get the relative error:

$$\frac{dV}{V} = \frac{0.0754}{8\pi(0.3)^2} = 0.03333 \qquad \frac{dx}{x} = \frac{0.01}{0.6} = 0.016$$

Chapter 4.4 Indeterminate forms and l'Hospital's Rule

$$\lim_{x\to 1}\frac{\ln x}{1-\sqrt{x}} \to \frac{1}{-\frac{1}{2}} = -2$$



L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Indeterminate forms

Recall: $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ (type $\frac{0}{0}$)

Similarly, the following are indeterminate forms:

- type $\frac{\infty}{\infty}$
- type $0\cdot\infty$
- type $\infty-\infty$
- type 1^{∞}
- type ∞^{0}
- type 0^{0}

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x)\ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

Obtain a common denominator to solve the limit further:

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)}$$

If we attempt to take the limit as x approaches zero from the right:

$$\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)}=\frac{e^0-1-0}{(0)(e^0-1)}=\frac{1-1-0}{(0)(0)}=\frac{0}{0}$$

The resulting form $0/0$ is indeterminate and L'Hospital's Rule can be applied. In order to apply L'Hospital's Rule, one must take the derivative of the top and the bottom (assuming it is in a form in which L'HR can be applied- it is in a form which is acceptable, that is: $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Therefore,

$$\lim_{x\to 0^+}\frac{(e^x-1-x)'}{(x(e^x-1))'}=\lim_{x\to 0^+}\frac{e^x-1}{(1)(e^x-1)+(x)(e^x)}=\lim_{x\to 0^+}\frac{e^x-1}{e^x-1+xe^x}$$

Now that L'HR has been applied, let us try to evaluate the limit once again, as x approaches 0 from the right (positive x values).

$$\lim_{x\to 0^+}\frac{e^x-1}{e^x-1+xe^x}=\frac{e^0-1}{e^0-1+(0)e^0}=\frac{1-1}{1-1+0}=\frac{0}{0}$$

Since the value was an indeterminate $\dfrac{0}{0}$ again, we can try to apply L'HR once more.

$$\lim_{x\to 0^+}\frac{(e^x-1)'}{(e^x-1+xe^x)'}=\lim_{x\to 0^+}\frac{e^x}{e^x+(x)e^x+(1)e^x}=\lim_{x\to 0^+}\frac{e^x}{2e^x+(x)e^x}$$

One more time, let us try to evaluate the limit as x approaches 0 from the right.

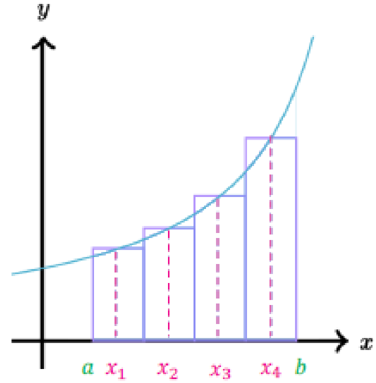
$$\lim_{x\to 0+}\frac{e^x}{2e^x+(x)e^x}=\frac{e^0}{2e^0+(0)e^0}=\frac{1}{2(1)+0}=\frac{1}{2}$$

Thus, after two applications of L'HR, the original limit was evaluated and found to have an actual value of one-half.

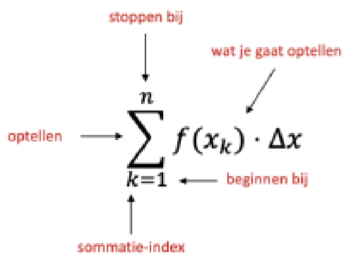
Week 3. Integration Techniques

Wiskunde B Refresher

- Riemannsommen
  - Oppervlakte $O(V)$ onder grafiek tussen $x = a$ en $x = b$ benaderen met n rechthoeken
    - Breedte $\Delta x = \frac{b-a}{n}$
  - $O(V) \approx f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x$
    $\approx \sum_{k=1}^{n} f(x_k) \cdot \Delta x$



  - Sommatieteken



  - Ondersom
    - Hoogte rechthoek: kleinste functiewaarde op het interval
  - Bovensom
    - Hoogte rechthoek: grootste functiewaarde op het interval

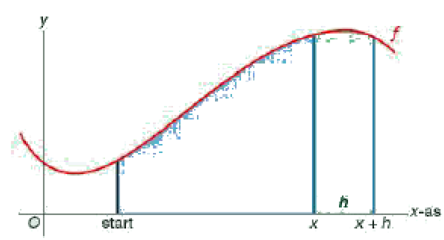
- Definitie integraal
  - $O(V) = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(x_k) \cdot \Delta x = \int_a^b f(x)\,\mathrm{d}x$

$$O(x+h) - O(x) \approx f(x) \cdot h$$

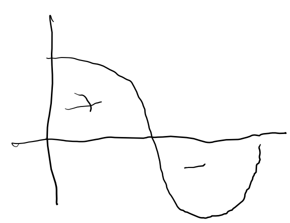
Links en rechts delen door h en limiet $h \to 0$ nemen

$$\lim_{h \to 0} \frac{O(x+h) - O(x)}{h} = \lim_{h \to 0} \frac{f(x) \cdot h}{h}$$

$$O'(x) = f(x)$$



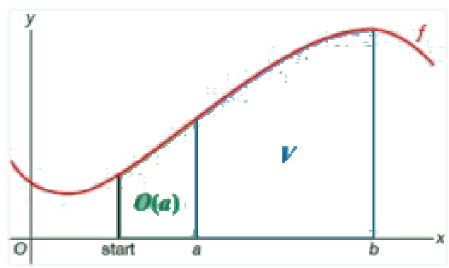
- Definitie primitieve
  - Een primitieve van $f(x)$ is een functie $F(x)$ waarvoor geldt $F'(x) = f(x)$
  - Notatie: $F(x)$ of $\int f(x)\,\mathrm{d}x$



Type functie	**Functie**	**Afgeleide**	**Primitieve**
Constante	$f(x) = a$	$f'(x) = 0$	$F(x) = ax + c$
Machtsfunctie	$f(x) = x^n$	$f'(x) = n \cdot x^{n-1}$	$F(x) = \frac{1}{n+1} \cdot x^{n+1} + c$ voor $n \neq -1$
	$f(x) = \frac{1}{x}$		$F(x) = \ln\lvert x\rvert + c$
Exponentiële functie	$f(x) = \mathrm{e}^x$	$f'(x) = \mathrm{e}^x$	$F(x) = \mathrm{e}^x + c$
	$f(x) = g^x$	$f'(x) = g^x \cdot \ln(g)$	$F(x) = \frac{1}{\ln(g)} \cdot g^x + c$
Goniometrische functie	$f(x) = \sin(x)$	$f'(x) = \cos(x)$	$F(x) = -\cos(x) + c$
	$f(x) = \cos(x)$	$f'(x) = -\sin(x)$	$F(x) = \sin(x) + c$

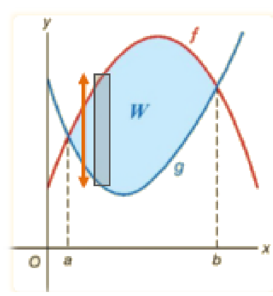
- Kettingregel bij primitiveren
  - Als $f(x)$ schrijven is als $f(u)$, waarbij $u(x)$ een lineaire functie is, dan is $F(x) = \frac{1}{u'(x)} \cdot F(u)$
    - Oftewel: $f(ax+b) \rightarrow F(x) = \frac{1}{a}F(ax+b)$

- Oppervlakte onder een grafiek
  - $$O(V) = \int_a^b f(x)\,\mathrm{d}x = O(b) - O(a) = F(b) + c - (F(a) + c) = F(b) - F(a)$$



- Hoofdstelling van de integraalrekening
  - $$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

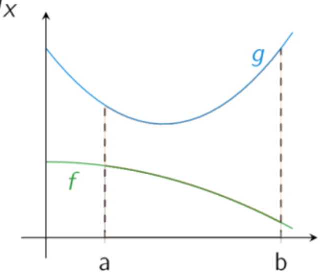
- Tussen grafieken
  - $$O(W) = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$$
    - Met $f(x) > g(x)$ op $[a, b]$



Properties of integrals

Suppose f and g are integrable. Then:

- $\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$
- $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx.$



- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

The Fundamental Theorem of Calculus

Theorem
Suppose $f(x)$ is continuous on $[a, b]$.
Then there exists a function $F(x)$ such that $f(x) = F'(x)$ on $[a, b]$.
For every such F we have:

$$\int_a^b f(x)dx = F(b) - F(a).$$

$\int_0^1 \sqrt{1-x^2}\,dx.$

Definition
Such function F is called a primitive function or anti-derivative of f.

Standard anti-derivatives

$\int \tan x\,dx = \ln|\sec x| + C$ $\mathbf{= -\ln|\cos(x)| + c}$

$$\int x^r dx = \tfrac{1}{r+1}x^{r+1} + C \text{ for } r \neq -1 \qquad \int e^x dx = e^x + C$$

$$\int \frac{1}{x}dx = \ln|x| + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\int \cos(x)dx = \sin(x) + C \qquad \int -\frac{1}{\sqrt{1-x^2}}dx = \arccos(x) + C$$

$$\int \sin(x)dx = -\cos(x) + C \qquad \int \frac{1}{1+x^2}dx = \arctan(x) + C$$

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

5.5 Substitution rule

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du, \text{ where } u = g(x).$$

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du$$

$$= \tfrac{2}{3}u^{3/2} + C = \tfrac{2}{3}(1+x^2)^{3/2} + C$$

$u'(x) = \frac{du}{dx}$

3 $$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

It is the counterpart to the chain rule for differentiation.

Always do a u-sub if you can; if you cannot, consider integration by parts.

A u-sub can be done whenever you have something containing a function (we'll call this g), and that something is multiplied by the derivative of g. That is, if you have $\int f(g(x))g'(x)dx$, use a u-sub.

Integration by parts is whenever you have two functions multiplied together--one that you can integrate, one that you can differentiate.

My strategy is to try to "play it out" in my mind and try to see which one will work better. The best way to get better at these sorts of integrals is to practice large sets of each type. Then, you start to think "Oh--this looks like a u-sub!" or, "maybe by-parts is better for this." **Practice is really the best way to get better at recognizing each type.**

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u, not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

For $f(x) = x^4 \sin x$, evaluate $f(-x)$

$$f(-x) = (-x)^4 \sin(-x)$$

$$= x^4 \sin(-x)$$

$$= -x^4 \sin x \qquad [\sin(-x) = -\sin x]$$

Because $f(-x) = -f(x)$, the function is odd and has rotational symmetry about the origin.

For an odd function $\int_{-a}^{a} f(x)\,dx = 0$ because of the symmetry. So then

$$\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx = 0$$

For an even function holds f(-x) = f(x). Then

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

7.1 Integration by parts

1

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Example: $\int x^4 e^{-x}\,dx = ?$

Define $I_n = \int x^n e^{-x}\,dx$.

Then

- $I_0 = -e^{-x} + C$;
- $I_n = -x^n e^{-x} + n\,I_{n-1}$ $\quad (n = 1, 2, 3, \ldots)$.

reduction formula

the reduction formula

7

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

- $I_n = -x^n e^{-x} + n\,I_{n-1}$ $\quad (n = 1, 2, 3, \ldots)$.

reduction formula

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

or

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

We can rearrange this equation as

1

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Formula 1 is called the **formula for integration by parts**. It is perhaps easier to remember in the following notation. Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, so, by the Substitution Rule, the formula for integration by parts becomes

2

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}\int x \sin x\,dx &= f(x)g(x) - \int g(x)f'(x)\,dx \\ &= x(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

$$\int x \sin x\,dx = (\sin x)\frac{x^2}{2} - \frac{1}{2}\int x^2 \cos x\,dx$$

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x\frac{dx}{x} \\ &= x\ln x - \int dx \\ &= x\ln x - x + C\end{aligned}$$

Suppose we want to evaluate the following integrals:

I. $\int x^2 e^x \, dx$

II. $\int \frac{e^x}{x} \, dx$

For which does integration by parts help?

A. for both;

✓**B.** only for I;

$$\int_1^\infty \frac{1}{x^p} dx \text{ is } \begin{cases} \text{convergent for } p > 1 \\ \text{divergent for } p \leq 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p} dx \text{ is } \begin{cases} \text{convergent for } p < 1 \\ \text{divergent for } p \geq 1 \end{cases}$$

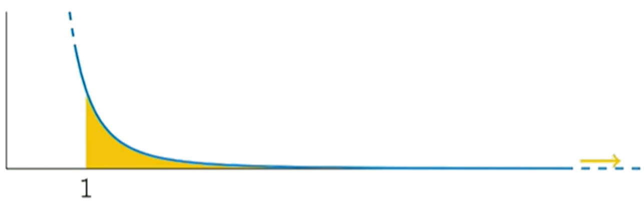
$$\int_0^\infty e^{ax} dx \text{ is } \begin{cases} \text{convergent for } a < 0 \\ \text{divergent for } a \geq 0 \end{cases}$$

$$\int_0^1 \ln(ax) dx \text{ is convergent for } a > 0$$

7.8 Improper Integrals

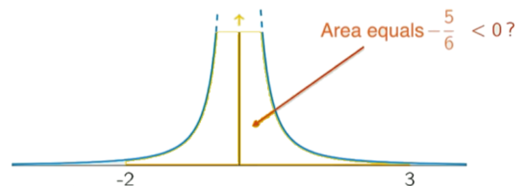
Type 1: when one of the terms is infinity. Type 2: When the integral goes through an undefined x

Type 1



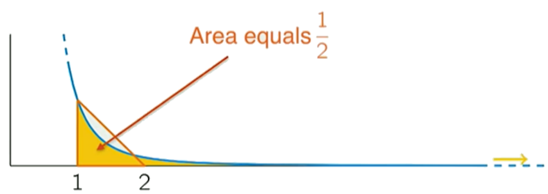
$$\int_1^\infty \frac{1}{x^3} \, dx$$

Type 2



$$\int_{-2}^{3} \frac{1}{x^2} \, dx = [-\frac{1}{x}]_{-2}^{3} = -\frac{1}{3} - (-\frac{1}{-2}) = -\frac{5}{6}$$

Work around, use limits:



$$\int_1^\infty \frac{1}{x^3} \, dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^3} \, dx = \lim_{t \to \infty} [-\frac{1}{2x^2}]_1^t = \lim_{t \to \infty} -\frac{1}{2t^2} + \frac{1}{2} = \frac{1}{2}$$

If the proper integral has an horizontal asymptote for type 1 (or it is just a perforation), the limit exists and it "converges". If it goes to + - inf for type 1 or it is a vertical asymptote for type 2, then it "diverges"

Theorem:

Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$, then:

1. $\int_a^\infty f(x) \, dx$ convergent $\Longrightarrow$ $\int_a^\infty g(x) \, dx$ convergent
2. $\int_a^\infty g(x) \, dx$ divergent $\Longrightarrow$ $\int_a^\infty f(x) \, dx$ divergent

Week 4.1 Sequences and series I

A sequence is a list of numbers written in a definite order. A sequence can be defined as a function whose domain is the set of positive integers. The notation a_n implicitly means f(n).

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

Notice that n doesn't have to start at 1. It can also be defined as function a : N -> R

EXAMPLE 1 Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \dfrac{n}{n+1}$ $\qquad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\qquad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\qquad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$ $\qquad \left\{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\right\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos\dfrac{n\pi}{6},\ n \geqslant 0$ $\qquad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$ ■

EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

If the signs of the terms alternate between positive and negatives we will need to multiply by a power of -1. $(-1)^n$ means we start with a negative term. $(-1)^{n+1}$ or $(-1)^{n-1}$ or $-(-1)^n$ means we start with a positive term if the first n=1.

(c) **The Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Each term is the sum of the two preceding terms. The first few terms are

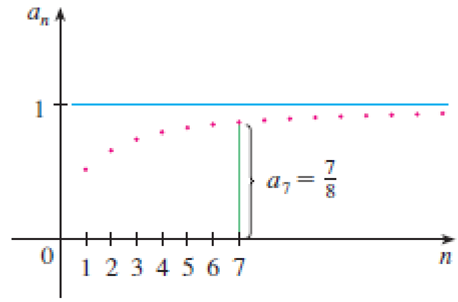
$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates.

In general, the notation

$$\lim_{n \to \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large.



All the previous limit laws hold for sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$

$$\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n$$

$$\lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n \qquad \lim_{n \to \infty} c = c$$

$$\lim_{n \to \infty} (a_n b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \quad \text{if } \lim_{n \to \infty} b_n \neq 0$$

$$\lim_{n \to \infty} a_n^p = \left[\lim_{n \to \infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Limits of sequences: rules of calculation

Substitution:
Let (a_n) be a sequence and f a function such that:

- $\lim_{n \to \infty} a_n = L$
- f is continuous in L

Then $\lim_{n \to \infty} f(a_n) = f(L)$.

Squeeze Theorem:
Let $(a_n), (b_n)$ and (c_n) be sequences such that:

- $\lim_{n \to \infty} a_n = L$
- $\lim_{n \to \infty} c_n = L$
- $a_n \leq b_n \leq c_n$ for all n sufficiently large.

Then $\lim_{n \to \infty} b_n = L$.

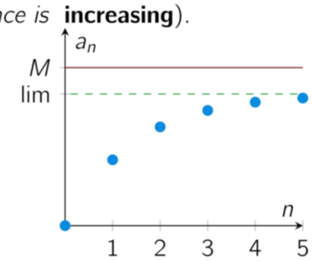
$$\text{If } \lim_{n \to \infty} |a_n| = 0, \text{ then } \lim_{n \to \infty} a_n = 0.$$

Monotone convergence theorem

Let (a_n) be a sequence such that:

- there is a number M such that $a_n \leq M$ for all n (*the sequence is bounded from* **above**),
- $a_{n+1} \geq a_n$ for all n (*the sequence is* **increasing**).

Then the sequence is convergent.

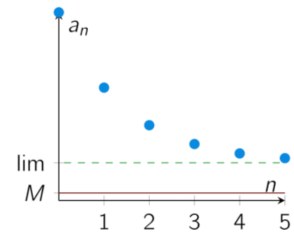


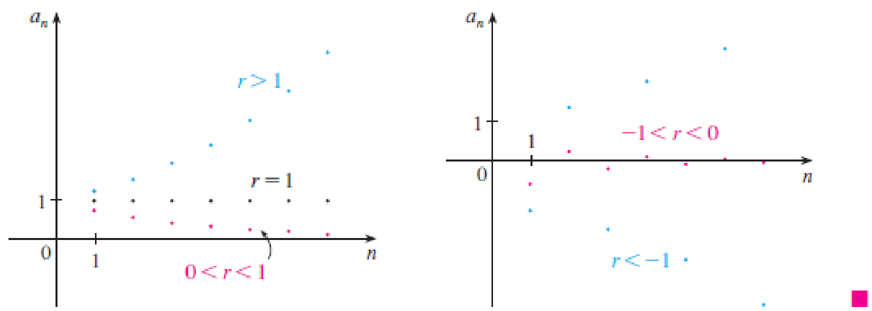
Monotone convergence theorem

Let (a_n) be a sequence such that:

- there is a number M such that $a_n \geq M$ for all n (*the sequence is bounded from* **below**),
- $a_{n+1} \leq a_n$ for all n (*the sequence is* **decreasing**).

Then the sequence is convergent.



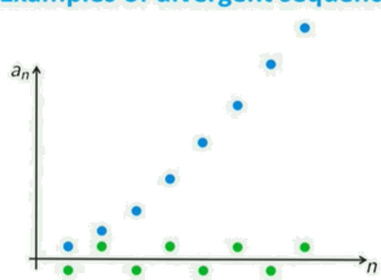


The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

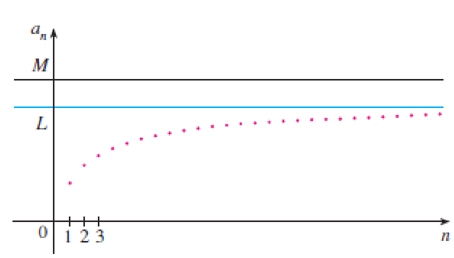
Examples of divergent sequences



Both, going to infinites and not settling at a specif value (i.e. sinus) makes the sequence divergent. (It does not converge to a specific value).

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.



3. Consider the following sequence:

$$\begin{cases} a_0 &= 2 \\ a_{n+1} &= \frac{a_n^2}{4} + \frac{1}{2} \end{cases}$$

Is this sequence convergent or divergent?
In case of convergence, find the limit.

Answer:
We claim that the following holds for all $n \in \mathbb{N}$: $0 \leq a_{n+1} \leq a_n$.
Let us check for $n = 0$:

$$0 \leq a_1 = \frac{3}{2} \leq a_0 = 2.$$

Suppose that for some $n \in \mathbb{N}$ we have $0 \leq a_{n+1} \leq a_n$. Then $0 \leq a_{n+1}^2 \leq a_n^2$, hence also $0 \leq \frac{a_{n+1}}{4} + \frac{1}{2} \leq \frac{a_n}{4} + \frac{1}{2}$. This implies that $0 \leq a_{n+2} \leq a_{n+1}$.

By induction we see that $0 \leq a_{n+1} \leq a_n$ is true for all $n \in \mathbb{N}$. In particular, this means that (a_n) is a decreasing sequence, bounded from below. By the Monotone Convergence Theorem, it is convergent.

Let's write $\lim_{n\to\infty} a_n = L$, then we should have $L = \frac{L^2}{4} + \frac{1}{2}$. That is: $L^2 - 4L + 2 = 0$. This equation has solutions $L = 2 \pm \sqrt{2}$. Since the sequence starts at 2 and is decreasing, we have $L = 2 - \sqrt{2}$.

4. Let $(a_n)_{n=1}^{\infty}$ be a sequence recursively defined as follows: $a_1 = 2$, $a_{n+1} = 4 - \dfrac{3}{a_n}$.
Given that this sequence is convergent, find $\lim\limits_{n\to\infty} a_n$.

Answer: 3
Let us call the limit L. The limit has to satisfy the recursion relation, so $L = 4 - \frac{3}{L}$. This has two solutions: $L = 1$ and $L = 3$.
The answers follows from the fact that this sequence is increasing. This can be shown by induction.

4. Consider the sequence (a_n) defined recursively as follows:

$$\begin{cases} a_1 = 2 \\ a_{n+1} = 5 - \frac{1}{3}a_n \text{ for all } n \in \mathbb{N} \end{cases}$$

Given that this sequence converges, find its limit.

Solution:
Let us call the limit L. The limit has to satisfy $L = 5 - \frac{1}{3}L$. Solving this equation gives $L = \frac{15}{4}$.

Series

Definition

Let $(a_1, a_2, a_3, \ldots)$ be a sequence of numbers.
The infinite sum

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called a series.

$1 + 2 + 3 + 4$

$\frac{2}{1} \neq \frac{3}{2} \neq \frac{4}{3} \neq$ not a common ratio

$2-1 = 3-2 = 4-3$ common difference = linear

$a_n = n \cdot (a_2 - a_1)$

Define the partial sums: $s_1 = a_1$

$$s_2 = a_1 + a_2$$
$$\vdots$$
$$s_N = \sum_{n=1}^{N} a_n$$

If $\lim_{N \to \infty} s_N$ exists and equals $L \in \mathbb{R}$, the series is convergent and we write: $\sum_{n=1}^{\infty} a_n = L$.

Otherwise, the series is **divergent**.

6 12 24 48

$\frac{12}{6} = \frac{24}{12} = \frac{48}{24}$ common ratio $r = \frac{a_2}{a_1}$

$a_n = r^n \cdot \frac{a_1}{r}$

$a_1 = 6$

$r = \frac{a_2}{a_1} = \frac{12}{6} = 2$

$a_n = 2^n \cdot \frac{6}{2} = 3 \cdot 2^n$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called a geometric series if there is a number r such that $\frac{a_{n+1}}{a_n} = r$ for all n.
This r is the common ratio of the series.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1-r} - \frac{a_1}{1-r} \cdot \lim_{n \to \infty} r^n$$

such series can be written as $\sum_1^{\infty} c \cdot r^n$

Theorem

Given a geometric series with common ratio r, we have:

- If $-1 < r < 1$, then the series is **convergent**.

  The sum is (*first term*) $\cdot \frac{1}{1-r}$.
- If $r \leq -1$ or $r \geq 1$, the series is **divergent**.

$r = \frac{a_2}{a_1}$

n^2 1 4 9 16 25

$4-1=3$ $9-4=5$ $16-9=7$ $25-16=9$

$5-3=2$ $7-5=2$ $9-7=2$

second difference is the power (second derivative)

2, 4, 8, 16, 32

2 4 8 16

if k and k+1 level differences are the same, then it could use geometric (try to find ratio)

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent. Then:

- $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

Theorem:

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

The (*logical*) *contrapositive* of this theorem is:

Corollary:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: if $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ need not be convergent!

Example: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent!

Definition:

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a *p*-series.

Theorem:

A p-series is

- Convergent if $p > 1$,
- Divergent if $p \leq 1$

Remark: the case $p = 1$ is called the harmonic series.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

2 **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

Theorem (Alternating Series Test)

If a series a_n satisfies

(i) It is alternating

(ii) $|a_{n+1}| \leq |a_n|$ for all n

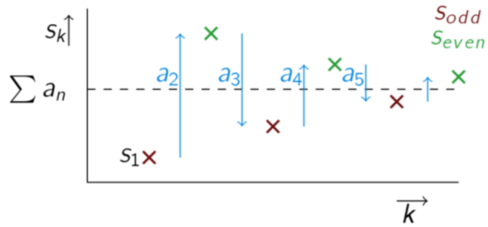
(iii) $\lim_{n\to\infty} |a_n| = 0$,

then the series is convergent.

not iff

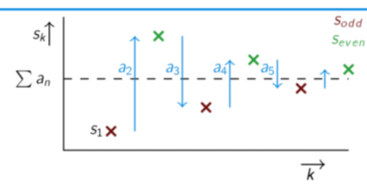
decreasing

a_n converges to zero (not necessarily $\sum a_n$)



Alternating series: error bound

Theorem
Suppose the series $\sum_{n=1}^{\infty} a_n$ satisfies the conditions of the Alternating Series Test and has sum s, then for each partial sum $s_N = \sum_{n=1}^{N} a_n$ the following holds: $|s - s_N| \leq |a_{N+1}|$



The sum error has to be smaller or equal to the next term

Comparison tests

Theorem:
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n,

   en $\sum a_n$ is also convergent.

2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n,

   then $\sum a_n$ is also divergent.

The limit comparison test

Theorem:
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Remember that : Error in the nth partial term is bounded by a_n+1

That is : Error in the 5th partial term is bounded by a_6

We need to find smallest n such that,

$$\frac{1}{n^6} < 0.00005$$

$$\frac{1}{n^6} < \frac{5}{100000}$$

Take reciprocal of both sides.

Since both the sides are positive, the Inequality reverses.

$$n^6 > \frac{100000}{5}$$

$$n^6 > 20000$$

Check By trial and error :

$$a_2 = 2^6 = 64$$

$$a_3 = 3^6 = 729$$

$$a_4 = 4^6 = 4096$$

$$a_5 = 5^6 = 15625$$

$$a_6 = 6^6 = 46656 \ (> 20000)$$

$n = 6$ is the smallest number for which this Inequality is satisfied.

Hence, $n = 6$ is the smallest number for which the error is less than 0.00005

We only need to add the first five terms of the series to approximate the sum within the allotted error.

ALTERNATING SERIES TEST

Suppose that we have the series $\sum a_n$, such that

$a_n = (-1)^n b_n$ OR $a_n = (-1)^{n+1} b_n$, Where $b_n \geq 0$ for all n

Then If the following two conditions are satisfied the series is convergent.

(1) $\lim_{n\to\infty} b_n = 0$

(2) b_n is a decreasing sequence

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$$

In the given problem $b_n = \sin\left(\frac{\pi}{n}\right)$

Which satisfies both the conditions of Alternating series.

1. $b_n = \sin\left(\frac{\pi}{n}\right)$ is decreasing.

It is because if $\frac{\pi}{n}$ is decreasing and $\sin x$ is increasing on $[0, \pi/2]$

Which Implies $\sin\left(\frac{\pi}{n+1}\right) < \sin\left(\frac{\pi}{n}\right)$ for $n > 2$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin(0) = 0$

Hence the given series is convergent by Alternating Series Test

Week 6. Sequences and Series II

11.6.1 Absolute convergence

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

> **1 Definition** A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Notice that if $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

Definition 1:
A series $\sum a_n$ is called **absolutely convergent**
if the series $\sum |a_n|$ is convergent.

Definition 2:
A series $\sum a_n$ is called **conditionally convergent**
if it is convergent but *not* absolutely convergent.

> **Theorem:**
> If a series is absolutely convergent, then it is convergent.

Remember that $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ is convergent

Therefore, the given series will be absolutely convergent when the following series converges.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+1}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1} < \sum_{n=1}^{\infty} \frac{1}{n^3}$$

p-series with $p = 3$ is converging

Since the absolute series is less than a converging series, it converges

Hence the given series is absolutely convergent

Ratios

n	1	2	3	4	5
$\frac{a_{n+1}}{a_n}$	0.66	0.50	0.44	0.42	0.40

$$\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\ldots=\sum_{n=1}^{\infty}\frac{n}{3^n}$$

$$a_n = \frac{n}{3^n} \qquad \frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}}\cdot\frac{3^n}{n} = \frac{n+1}{3n}$$

Comparison theorem

$$\frac{a_{n+1}}{a_n} \leq 0.7 \Rightarrow a_{n+1} \leq 0.7a_n$$

$$a_n \leq \frac{1}{3}(0.7)^{n-1}$$

$$a_2 \leq 0.7a_1$$

$$a_3 \leq 0.7a_2 \leq (0.7)^2 a_1$$

$$a_4 \leq 0.7a_3 \leq (0.7)^3 a_1$$

"The terms of our original series are smaller than the terms of a convergent geometric series". Since the geometric is convergent and the original is smaller than the convergent, the original is also convergent.

11.6.2 Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \neq 0$ for all n. Suppose $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = L$

Then we have:

- If $L < 1$,
  then the series is **absolutely convergent** (and hence convergent).
- If $L > 1$ or $L = \infty$,
  then the series is **divergent**.
- If $L = 1$ or if the limit does not exist,
  then the test is inconclusive.

Example:

$$a_1 = 1, \quad a_{n+1} = \frac{2+\cos n}{\sqrt{n}}a_n$$

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{2+\cos n}{\sqrt{n}}a_n}{a_n}\right| = \left|\frac{2+\cos n}{\sqrt{n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{2+\cos n}{\sqrt{n}}\right| = 0$$

The $(2+\cos n)$ oscillates between 1 and 3, while $\sqrt{n}$ goes to ∞. The limit is < 1, so it is absolutely convergent and therefore convergent.

11.8 Power series

Definition

A series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

is called a <u>power series</u> in $(x-a)$.

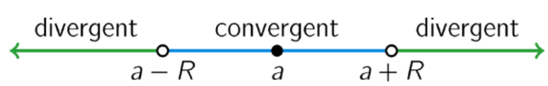
Convergence may depend on x

Power series: convergence

Theorem

There is an $R \geq 0$, possibly ∞, such that:

- The series is (absolutely) convergent for $|x-a| < R$
- The series is divergent for $|x-a| > R$



R is the **radius of convergence**,
a is the **center of convergence**.

Finding the radius

To find the Radius of convergence, we first find the Interval of convergence.

Once we know that the Interval of convergence is (m, n)

Then the Radius of convergence is given by $R = \dfrac{n-m}{2}$

The interval of convergence is the domain of x values for which the power series converges. It is found by using the ratio test to determine for what values the series is convergent. The interval endpoints must be analyzed, sometimes they converge, sometimes they don't.

4 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Finding the radius example

$\sum_{n=1}^{\infty}(-1)^n n x^n$

Let $a_n = (-1)^n n x^n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{(-1)^n n x^n}\right|$$

$$= \left|\frac{(n+1)x}{n}\right|$$

$$= (1+\frac{1}{n})|x|$$

Take the limit for the Ratio test

$$\lim_{n\to\infty}\left[(1+\frac{1}{n})|x|\right] = |x|$$

Converges if $|x| < 1$, so the radius of convergence is $R = 1$ and the interval is from -1 to 1.

Check the endpoints of the interval. $x = -1$:

$\sum_{n=1}^{\infty}(-1)^n n(-1)^n = \sum_{n=1}^{\infty}((-1)(-1))^n n = \sum_{n=1}^{\infty} 1^n n = \sum_{n=1}^{\infty} n$

This will approach ∞ and diverge.

For $x = 1$:

$\sum_{n=1}^{\infty}(-1)^n n(1)^n = \sum_{n=1}^{\infty}(-1)^n n$

The limit as $n \to \infty$ doesn't exist, so this diverges also.

The endpoints are not included in the interval of convergence.

$(-1, 1)$

Ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right| = |2x+1|\lim_{n\to\infty}(n+1).$$

Note that this limit is infinite for $2x+1 \neq 0$. So the power series only converges for $x = -\frac{1}{2}$.

So, the radius of convergence for this power series is $R = 0$.

Function representation of series

A geometric series

$$1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

Convergent if and only if $-1 < x < 1$

If $-1 < x < 1$ then $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Approximation by polynomials

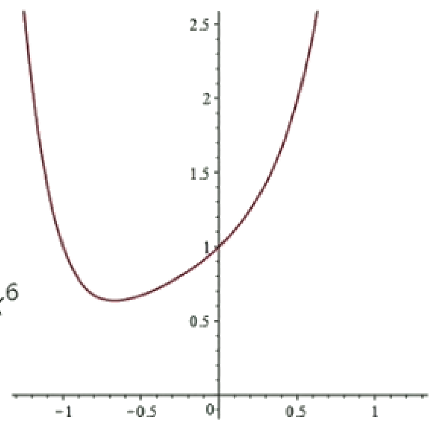
If $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ then $f(x) \approx s_n$

$$s_n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$$

A geometric series

$s_1 = 1 + x$
$s_2 = 1 + x + x^2$
$s_3 = 1 + x + x^2 + x^3$
$s_4 = 1 + x + x^2 + x^3 + x^4$
$s_5 = 1 + x + x^2 + x^3 + x^4 + x^5$
$s_6 = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$

$$\boxed{\lim_{n\to\infty} s_n = \frac{1}{1-x} \text{ if } -1 < x < 1}$$



2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\displaystyle\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

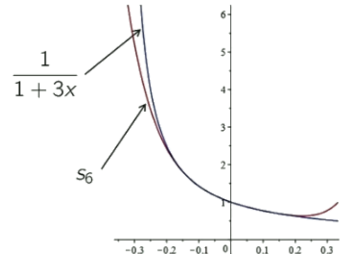
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Approximation by polynomials

$$s_6 = \sum_{i=0}^{6}(-3)^i x^i = 1 - 3x + 9x^2 - 27x^3 + 81x^4 - 243x^5 + 729x^6$$

$$\boxed{\frac{1}{1+3x} \approx s_6}$$



Finding the power series

$$\frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum_{n=0}^{\infty}(-3x)^n = \sum_{n=0}^{\infty}(-3)^n x^n$$

$$-1 < (-3x) < 1 \Leftrightarrow -\frac{1}{3} < x < \frac{1}{3}$$

Finding the power series

$$\frac{1}{5-x} = \frac{1}{5}\cdot\frac{1}{1-\frac{x}{5}} = \frac{1}{5}\cdot\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\frac{1}{5^{n+1}}x^n$$

$$-1 < \frac{x}{5} < 1 \Leftrightarrow -5 < x < 5$$

$$\frac{1}{5-x} = \frac{1}{1-(x-4)} = \sum_{n=0}^{\infty}(x-4)^n$$

$$-1 < (x-4) < 1 \Leftrightarrow 3 < x < 5$$

Differentiating and integrating power series

Theorem

Suppose $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$.

Then:

- $f'(x) = c_1 + 2c_2(x-a) + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
- $\int f(x)dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \ldots$
  $= C + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$.

The radius of convergence remains the same!

Examples of functions as power series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1 < x < 1)$
- $\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \qquad (-1 < x \le 1)$
- $\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \qquad (-1 \le x \le 1)$

Since $\sum_{n=1}^{\infty} nc_n x^{n-1}$ is derivative of $\sum_{n=0}^{\infty} c_n x^n$, it has the same radius of convergence, that is 10.

Note that the sum of the geometric series with Initial term a and Common ratio r is

$$S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

The given function can be interpreted as

$$\frac{5}{1-(4x^2)} = \frac{a}{1-r}$$

Therefore, we can say that $f(x)$ is a sum of a geometric series with Initial term $a = 5$ and common ratio $r = 4x^2$

Therefore,

$$f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (5)\left(4x^2\right)^n$$

This is the power series representation of $f(x)$

We know that the geometric series converges when $|r| = \left|4x^2\right| < 1$

$$x^2 < \frac{1}{4}$$

Interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Radius of convergence is $\frac{1}{2}$

Why are we not checking for the end points of the interval of convergence?

This is because we know that a geometric series is divergent when $|r| = 1$. We have to check for the end points of the interval of convergence when using the Limit/Root test, this is because they do not provide any information when their signature limits are 1. But here we are using a known/popular result, which follows from the following:

$$\lim_{k\to\infty} \sum_{n=0}^{n=k} ar^n = \lim_{k\to\infty} \frac{a\left(1-r^{k+1}\right)}{1-r}$$

Note that the sum of the geometric series with Initial term a and Common ratio r is

$$S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$f(x) = \frac{x^2}{x^4 + 16}$$

Divide both numerator and denominator by 16

Note that the given function can be interpreted as

$$\frac{\frac{x^2}{16}}{\frac{x^4}{16}+1} = \frac{\frac{x^2}{16}}{1-\left(-\frac{x^4}{16}\right)} = \frac{a}{1-r}$$

Therefore we can say that $f(x)$ is a sum of a geometric series with Initial term $a = \frac{x^2}{16}$ and common ratio $r = -\frac{x^4}{16}$

Therefore,

$$f(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{x^2}{16}\right)\left(-\frac{x^4}{16}\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^{n+1} x^{4n+2}$$

This is the power series representation of $f(x)$

We know that the geometric series converges when $|r| = \left|-\frac{x^4}{16}\right| < 1$

$$\frac{x^4}{16} < 1$$

$$x^4 < 16$$

$$x^4 < 2^4$$

Interval of convergence is $(-2, 2)$

Radius of convergence is 2

Use differentiation to find power series for f(x) = 1/(1+x)^2

We know the power series :

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

Replace t with $-x$ to get the power series for $\frac{1}{1+x}$

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$$

Differentiate both sides with respect to x

$$\frac{d}{dx}\left[\frac{1}{1+x}\right] = \sum_{n=1}^{\infty} \frac{d\left[(-x)^n\right]}{dx}$$

Note that the summation is starting from $n = 1$ because of the derivative of the first term was 0

$$\frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} \underbrace{n(-x)^{n-1} \times (-1)}_{\text{Using Chain Rule}}$$

Now Multiply both sides by -1

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} n(-x)^{n-1}$$

In the given problem

$$a_n = (-1)^n\,(n+1)\,x^n$$

And

$$a_{n+1} = (-1)^{n+1}\,(n+2)\,x^{n+1}$$

Therefore

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(n+2)x^{n+1}}{(-1)^n(n+1)x^n} = \frac{-(n+2)x}{n+1}$$

And

$$L = \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{-(n+2)x}{n+1} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} |x|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n+1} + \frac{1}{n+1} \right) |x|$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n+1} \right) |x|$$

$$= \left(1 + \frac{1}{\infty} \right) |x|$$

$$= (1+0)\,|x| = |x|$$

The series will converge when $L < 1$

That is when

$$|x| < 1$$

Hence the radius of convergence is 1

Indefinite integral as power series

$$\int \frac{t}{1-t^8}\, dt$$

This is already in the form $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we just have to take the top t off to the side, then put it back in.

$$\frac{t}{1-t^8} = t \cdot \frac{1}{1-(t^8)} = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1}$$

Now integrate the series.

$$\int \left(\sum_{n=0}^{\infty} t^{8n+1}\right)\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$$

R for the original series and the integral are the same, so we can use the original to find R.

$$a_n = t^{8n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{t^{8(n+1)+1}}{t^{8n+1}} = \frac{t^{8n+9}}{t^{8n+1}} = t^8$$

Since all the n are gone from the expression, it doesn't matter what happens to n

$$\lim_{n\to\infty} |t^8| = |t^8|$$

The series will converge when $|t^8| < 1$, or $|t| < 1$. So $R = 1$.

Start with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$.

Then we have: $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

and therefore: $\frac{x^3}{1+x^2} = x^3 - x^{3+2} + x^{3+4} - x^{3+6} + \ldots.$

Hence: $\int_0^{\frac{1}{3}} \frac{x^3}{1+x^2} dx \approx \left[\frac{x^{3+1}}{3+1} - \frac{x^{3+2+1}}{3+2+1} + \frac{x^{3+4+1}}{3+4+1} - \frac{x^{3+6+1}}{3+6+1} \right]_0^{\frac{1}{3}}.$

Taylor and Maclaurin series

Definition

Let f be a function for which all derivatives exist in some point a.

The Taylor series of f at a is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Here $f^{(n)}(a)$ is the n-th derivative of f in a.

Remark

For $a = 0$ the series is also called the **Maclaurin series** of f.

Theorem

If f is equal to a power series near a, then

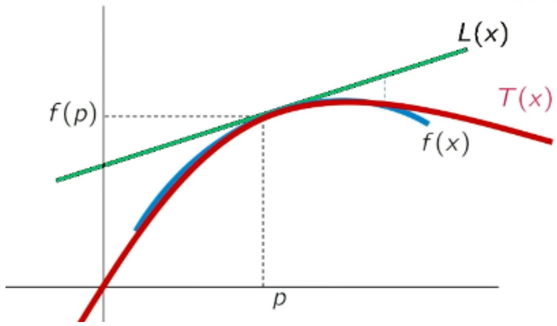
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on an open interval containing a.

If this holds, the function is called analytic at a.

Approximating a function

$$T(x) = A + B(x-p) + C(x-p)^2$$



$$\begin{aligned} &(1) \quad f(p) = T(p) \\ &(2) \quad f'(p) = T'(p) \\ &(3) \quad f''(p) = T''(p) \end{aligned}$$

Finding the coefficients of the parabola

$$\begin{aligned} T(x) &= A + B(x-p) + C(x-p)^2 \\ T'(x) &= B + 2C(x-p) \\ T''(x) &= 2C \end{aligned}$$

$$\begin{aligned} &(1) \quad f(p) = T(p) \implies A = f(p) \\ &(2) \quad f'(p) = T'(p) \implies B = f'(p) \\ &(3) \quad f''(p) = T''(p) \implies C = \frac{1}{2}f''(p) \end{aligned}$$

$$T(x) = f(p) + f'(p)(x-p) + \frac{1}{2}f''(p)(x-p)^2$$

$$T(x) = \qquad L(x) \qquad + \frac{1}{2}f''(p)(x-p)^2$$

Shipping container

$$\text{height} = x = \sqrt[3]{11}$$

$$p = 8,\ \sqrt[3]{8} = 2 \qquad T_2(x) = f(8) + f'(8)(x-8) + \frac{1}{2}f''(8)(x-8)^2$$

$$\begin{aligned} f(x) &= \sqrt[3]{x} & f(8) &= \sqrt[3]{8} = 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{3}8^{-2/3} = \frac{1}{12} \\ f''(x) &= -\frac{2}{9}x^{-5/3} & f''(8) &= -\frac{2}{9}8^{-5/3} = \frac{-1}{9 \cdot 16} \end{aligned}$$

$$\begin{aligned} f(11) &\approx f(8) + f'(8)(11-8) + \frac{1}{2}f''(p)(11-8)^2 \\ &= 2 + \frac{1}{4} - \frac{1}{32} \qquad \implies \sqrt[3]{11} \approx 2.21875 \end{aligned}$$

Binomial series

Theorem:

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \text{ for } -1 < x < 1,$$

where the binomial coefficients are given by:

$$\binom{r}{0} = 1, \quad \binom{r}{1} = r, \quad \ldots, \quad \binom{r}{n} = \frac{r \cdot \ldots \cdot (r-n+1)}{n!}$$

Some standard series

You don't have to know these by heart.

$\frac{1}{1-x}$	$= \sum_{n=0}^{\infty} x^n$	$= 1 + x + x^2 + x^3 + \ldots$	$R = 1$
e^x	$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$	$R = \infty$
$\sin(x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$	$R = \infty$
$\cos(x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$	$R = \infty$
$\arctan(x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$	$R = 1$
$\ln(1+x)$	$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$	$R = 1$
$(1+x)^r$	$= \sum_{n=0}^{\infty} \binom{r}{n} x^n$	$= 1 + rx + \frac{r(r-1)}{2!} x^2 + \ldots$	$R = 1$

Use Taylor series to evaluate the limit.

$$\lim_{x\to 0} \frac{\cos(x) - 1}{e^x - 1 - x} =$$

Use the Taylor series for $\cos(x)$ and divide by the Taylor series of e^x. Subtract -1 above and subtract $-1-x$ below.

$$\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - 1}{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots - 1 - x} = \frac{-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots}{\frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

$x \to 0$ so $\lim_{x\to 0} \frac{\cos(x) - 1}{e^x - 1 - x} = -1$

Use a Taylor series to evaluate the limit

$$\lim_{x\to 0} \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ln(1+x)}{x^5} =$$

You skipped this question. The correct answer is $-\frac{1}{5}$.

The Taylor series for $\ln(1+x)$ is $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

This leads to $\lim_{x\to 0} -\frac{1}{5} + \frac{1}{6}x - \dots = -\frac{1}{5}$

The Maclaurin series of $e^{-2x^3} = 1 + ax^3 + bx^6 + cx^9 + \dots$.

Determine a.

You skipped this question. The correct answer is $a = -2$.

Use the Maclaurin series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$

For e^x the Maclaurin series is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Filling in $-2x^3$ leads to

$e^{-2x^3} = 1 - 2x^3 + bx^6 + cx^9 + \dots$ so $a = -2$.

b=2, c=-4/3

Use Taylor series to evaluate the limit.

$$\lim_{x\to 0} \frac{\arctan(x) - x}{x^3} =$$

You skipped this question. The correct answer is $-\frac{1}{3}$.

Use the Taylor series for $\arctan(x)$. Substract the x and divide by x^3.

$$\frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots - x}{x^3} = -\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \ldots$$

$x \to 0$ so $\lim_{x\to 0} \frac{\arctan(x) - x}{x^3} = -\frac{1}{3}$

Use the first four non-zero terms of the Taylor series of $\sin(x^2)$ at $x = 0$ to find an approximation of

$$\int_0^1 \sin(x^2)\, dx \approx$$

Round your answer to 2 decimal places!

The Taylor series of the function $\sin(x^2)$ is:

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$$

Hence we have: $\int_0^1 \sin(x^2)\, dx \approx$

$$\left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \ldots\right]_0^1$$

Determine the coefficients a_n of the power series $\sum_{n=0}^{\infty} a_n x^n$ of $\frac{e^{1x} - 1}{x}$.

The correct answer is $\frac{(1)^{n+1}}{(n+1)!}$.

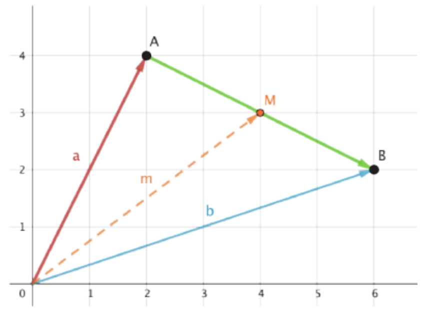
$a_n =$

The solution can be found by substituting $1x$ for x in the Taylor series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ of the exponential function. Subtract -1 and divide by x to find the coefficients.

Week 7. Multivariate functions and differentiation

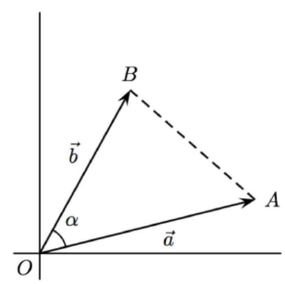
Vectors recap

Vectormeetkunde

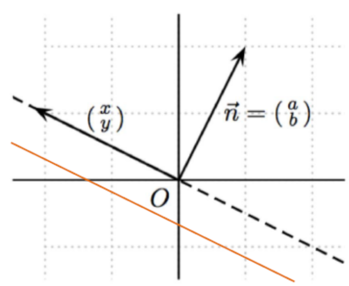


- $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$
  - Plaatsvector $\vec{a} = \overrightarrow{OA} = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$
  - Lengte: $|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2}$
- Berekeningen
  - Optellen: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$
  - Vermenigvuldigen met getal: $k \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} k \cdot a_1 \\ k \cdot a_2 \end{pmatrix}$
- Vectoren in figuren
  - $\overrightarrow{AB} = \vec{b} - \vec{a}$
  - Lengte $AB = |\vec{b} - \vec{a}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$
  - $\vec{m} = \frac{1}{2}(\vec{a} + \vec{b})$

Vectoren en hoeken

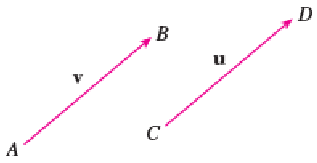


- Hoek tussen vectoren
  - $\cos(\alpha) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$
  - Inproduct: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$



- Onderling loodrechte vectoren
  - $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0$
  - Normaalvector van een lijn staat loodrecht op die lijn
    - $\vec{n}_k = \begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow k: ax + by = c$
    - Vectorvoorstelling ↔ vergelijking

Other notations: We denote a vector by printing a letter in **boldface**.



For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector** $\mathbf{v}$, shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$. Notice that the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$ even though it is in a different position. We say that $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$. The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Numbers that multiply (scale) a vector are called 'scalar'.

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Operations in 2D

Suppose $\mathbf{x} = \langle x_1, x_2 \rangle$ and $\mathbf{y} = \langle y_1, y_2 \rangle$.

Then

- $\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, x_2 + y_2 \rangle$ (addition)
- $c\,\mathbf{x} = \langle cx_1, cx_2 \rangle$ (scalar multiplication)
- $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ (norm)
- $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$ (dot product)

Special notation: $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$.

The dot product of the same vector (i.e. $\mathbf{x}^2$) is equal to the norm (length) squared.

Operations in 3D

Suppose $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ and $\mathbf{y} = \langle y_1, y_2, y_3 \rangle$.

Then

- $\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3 \rangle$ (addition)
- $c\,\mathbf{x} = \langle cx_1, cx_2, cx_3 \rangle$ (scalar multiplication)
- $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (norm)
- $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ (dot product)

Special notation: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

The dot product and angles

Theorem
Let **a**, **b** be vectors in $\mathbb{R}^n$.
Let θ be the angle between the vectors.
Then:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos(\theta).$$

Corollary
For $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ we have:

$$\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \text{ orthogonal to } \mathbf{b}$$

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Iff $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$ or $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$ then a and b are parallel (also iff $a_x/b_x = a_y/b_y$)

Some special subsets

In $\mathbb{R}^2$

$ax + by = c$	A line orthogonal to $\langle a, b \rangle$
$(x-a)^2 + (y-b)^2 = r^2$	The circle with center (a, b) and radius r

In $\mathbb{R}^3$

$ax + by + cz = d$	A plane orthogonal to $\langle a, b, c \rangle$
$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$	The sphere with center (a, b, c) and radius r

unit vectors are vectors with length 1.

Inner product = dot product

It is given that $\mathbf{r} = 2 \cdot \mathbf{u} + 3 \cdot \mathbf{v}$. Furthermore, it is given that: $|\mathbf{u}| = 3$, $|\mathbf{v}| = 2$ and the dot product $\mathbf{u} \cdot \mathbf{v} = -2$.

Determine the length of vector $\mathbf{r}$. Give the exact value.

Use $|\mathbf{r}| = \sqrt{(2\,|\mathbf{u}|)^2 + (3\,|\mathbf{v}|)^2 + 2 \cdot \mathbf{u} \cdot \mathbf{v} \cdot 2 \cdot 3} = 4\sqrt{3}.$

dot product

It holds that $\mathbf{r} = -2 \cdot \mathbf{u} + 3 \cdot \mathbf{v} + 1 \cdot \mathbf{w}$. Furthermore, it is given that the three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are *unit vectors* that are perpendicular to each other.

Determine the length of vector $\mathbf{r}$. Give the exact value.

In this exercise it holds that $\mathbf{u} \cdot \mathbf{u} = 1$ and $\mathbf{u} \cdot \mathbf{v} = 0$, etc. The axis along the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other and these vectors have length 1.

$$|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = (-2 \cdot \mathbf{u} + 3 \cdot \mathbf{v} + 1 \cdot \mathbf{w}) \cdot (-2 \cdot \mathbf{u} + 3 \cdot \mathbf{v} + 1 \cdot \mathbf{w}) = (-2)^2 + (3)^2 + (1)^2$$

$$\Longrightarrow |\mathbf{r}| = \sqrt{(-2)^2 + (3)^2 + 1^2} = \sqrt{14}\,.$$

Functions of several variables

Definition A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

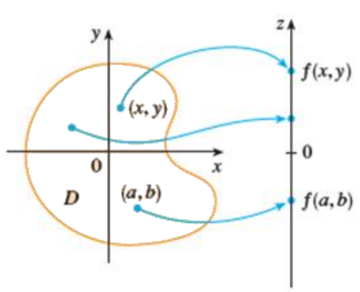


FIGURE 1

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

A function of two variables is just a function whose domain is a subset of $\mathbb{R}^2$ and whose range is a subset of $\mathbb{R}$. One way of visualizing such a function is by means of an arrow diagram (see Figure 1), where the domain D is represented as a subset of the xy-plane and the range is a set of numbers on a real line, shown as a z-axis. For instance, if $f(x, y)$ represents the temperature at a point (x, y) in a flat metal plate with the shape of D, we can think of the z-axis as a thermometer displaying the recorded temperatures.

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

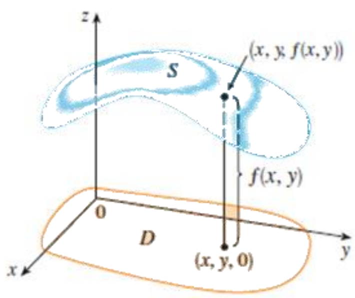


FIGURE 5

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

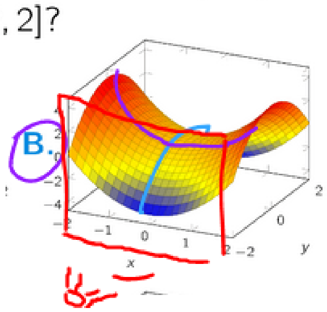
Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

Trick: Make one of the other variables a constant (i.e 0,1) and analyze the behavior of the other variable.

Functions of two variables

$D = [-2,2] \times [-2,2]$

Which figure shows the graph of $f(x, y) = x^2 - y^2$ on the domain $[-2, 2] \times [-2, 2]$?



Take $x = 0$:

$f(0, y) = -y^2$

Take $y = 0$:

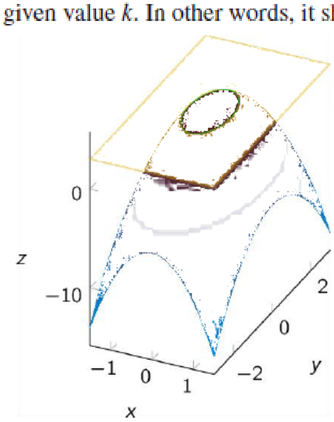
$f(x, 0) = x^2$

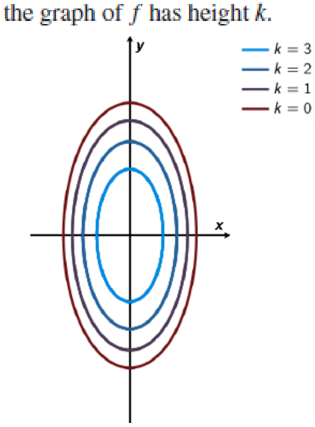
Contour map

A plot with several level curves

Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

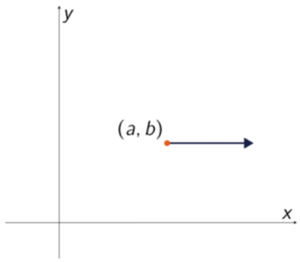
A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.





Partial derivatives

Partial derivative in x-direction

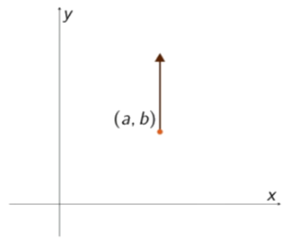


$p(x) = f(x, b)$

$f_x(a, b) = p'(a)$

$$\mathbf{f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}}$$

Partial derivative in y-direction



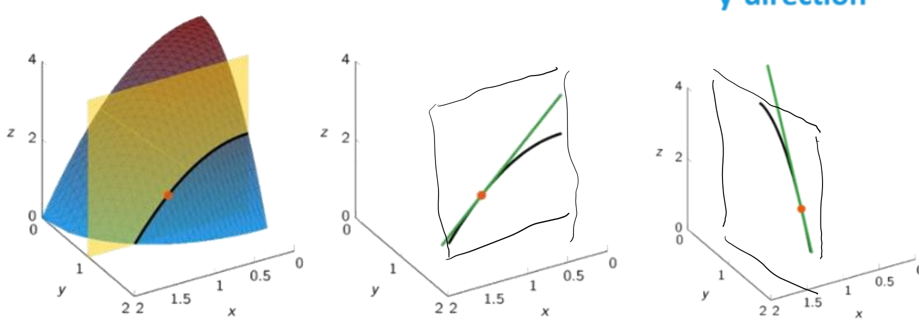
$q(y) = f(a, y)$

$f_y(a, b) = q'(b)$

$$\mathbf{f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

Partial derivative in x-direction

y-direction



Therefore, for $f(x, y) = x^3 + x^2y^3 - 2y^2$ we have $f_x(x, y) = 3x^2 + 2xy^3$ since y is a constant.

$$\text{Alternative notation:} \quad f_x(x,y) = \frac{\partial f}{\partial x}(x,y), \quad f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$$

The del(ta) symbol of the partial derivative is different than of the derivative as they are different things.

When doing higher partial derivatives we read from left to right:

Higher partial derivatives

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

Example:

$$f(x, y) = \sin(xy)$$

$f_x(x, y) = y\cos(xy)$ $\quad f_y(x, y) = x\cos(xy)$

$f_{xx}(x, y) = -y^2\sin(xy)$ $\quad f_{yy}(x, y) = -x^2\sin(xy)$

$f_{xy}(x, y) = \cos(xy) - xy\sin(xy)$

$f_{yx}(x, y) = \cos(xy) - xy\sin(xy)$

Clairaut's theorem

Theorem:
Suppose that f is defined on a disk D that contains the point (a, b).
If the functions f_{xy} and f_{yx} are both continuous on D, then

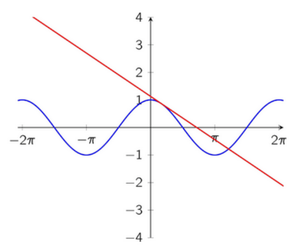
$$f_{xy}(a, b) = f_{yx}(a, b)$$

(All the functions in CSE1200 satisfy this condition) (in practice it is often too)

Aso fxxy = fxyx. So it's about the count, not the order.

Tangent planes

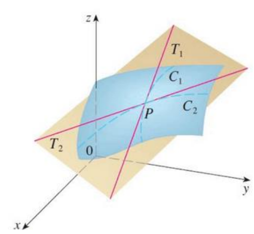
Tangent line



$$y = f(a) + f'(a)(x - a)$$

Tangent line is also equal to the first order (x^0 and x^1) taylor polynomial.

Tangent plane



The blue (middle) part represents the graph of the function of two variables. We can take the intersection of 2 curves and calculate the tangent lines T1 and T2 for each curve respectively. Then those lines make the triangles of the (flat) tangent plane to the graph.

We use partial derivatives at P for curves deliberately parallel to the x and y axis respectively so that:

Tangent plane and linearization

Theorem:
Suppose f has continuous partial derivatives. The tangent plane to the surface $z = f(x, y)$ at the point $P = (a, b, f(a, b))$ can be described by the equation:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The function L(x,y) = z is called the linearization (best 2D linear approximation for the 3D graph) of f at the point (a,b).

$f(x, y) = \ln(x^2 - y)$.
Which of the following equations describes the tangent plane to the graph at $(2, 1, \ln(3))$?

✔**A.** $z = \ln(3) + \frac{4}{3}(x-2) - \frac{1}{3}(y-1)$

$$f_x(x, y) = \frac{2x}{x^2 - y}$$

$$f_x(2, 1) = \frac{4}{4-1} = \frac{4}{3}$$

$$f_y(x, y) = \frac{-1}{x^2 - y}$$

$$f_y(2, 1) = \frac{-1}{4-1} = -\frac{1}{3}$$

Linearization

Use a linearization to estimate the distance of the point $(1.01, 1.94, 2.01)$ to the origin.

✔**B.** 2.97

Distance of the point (x, y, z) to the origin:

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Linearization at the point $(1, 2, 2)$:

$$L(x, y, z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$$

$$L(1.01, 1.94, 2.01) = 3 + \frac{0.01}{3} - \frac{0.12}{3} + \frac{0.02}{3} = 2.97$$

The equation of state for the *van der Waals gas in general* is equal to $\left(p + \frac{a}{v^2}\right)(v - b) = RT$ with pressure p, molar volume v, temperature T and gas constant $R = 8$.

The numbers a and b are specific for the gas which is considered. In the above example $a = 2$ and $b = 1$. The relative volume expansion coefficient is defined by $\alpha_v = \frac{1}{v}\frac{\partial v}{\partial T}$.

Remark
With this formula of the expansion coefficient, one can devise some experiments and from the measurements of this coefficient one can calculate the numbers a and b.

Derive an expression for the relative volume expansion α_V in terms of a, b, v, T and R. If necessary use the van der Waals equation to remove any instance of p.

$$\alpha_V = \frac{R(v-b)}{RTv - \frac{2a}{v^2}(v-b)^2}.$$

The directional derivative

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

(formal definition, rarely used)

Easier way to compute:

$$D_u f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2$$

where u is the unit vector <u_1, u_2> that points to the direction from which we are interested to know its slope. This equals the dot product.

$$= \underbrace{\langle f_x(x, y), f_y(x, y) \rangle}_{\text{Gradient}} \cdot \mathbf{u}$$

Definition:
The gradient ∇f of a function f at (x, y) is given by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The direction vector **u** must be formatted to an actual unit vector (a vector of length 1). I.e. for <3,1> do:

$$|\langle 3, 1 \rangle| = \sqrt{9 + 1} = \sqrt{10}$$

$$\text{So } \mathbf{u} = \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$$

(unit vector conversion)

Properties of the gradient

- $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = |\nabla f(x, y)|\,|\mathbf{u}| \cos(\theta)$,

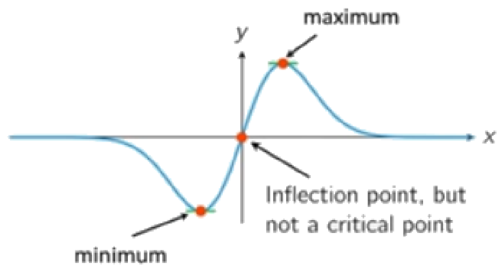
  where θ is the angle between $\nabla f(x, y)$ and $\mathbf{u}$.
- $\nabla f(x, y)$ gives the direction in which the derivative is maximal.
- $|\nabla f(x, y)|$ is the maximal value of the derivative at (x, y).
- ∇f is perpendicular to the level curve.

The minimal is the maximal * -1

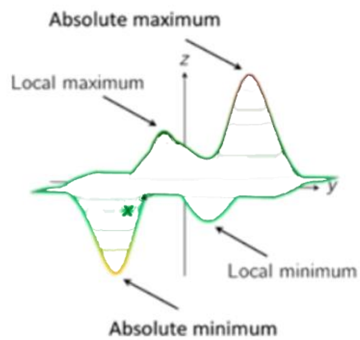
Week 8. Multivariate functions: differentiation and integration

Critical Points: Maximum and minimum values

For 1 variable



For 2 variables



Property of local extreme values

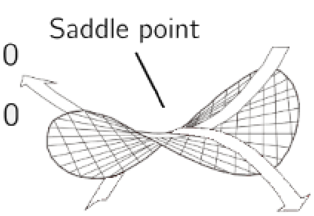
Theorem
If f has a local maximum or minimum at a point (a, b) and the first order partial derivatives exist at that point, then the first order partial derivatives at that point are equal to 0.
In a formula:

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

$$\mathbf{z = x^2 - y^2}$$

$$f_x(x, y) = 2x \implies f_x(0, 0) = 0$$

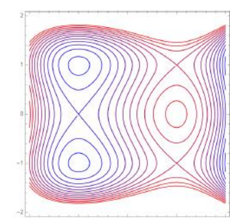
$$f_y(x, y) = -2y \implies f_y(0, 0) = 0$$



Saddle points

How many saddle points does this function have on the given region?
Note: blue and red are respectively the low and high values of the function

A. 0
B. 1
C. 2
✓**D.** 3

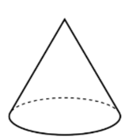


Note that for a minimum in a 2 variable function it must hold that if we had an imaginary marble, it would need to reach a stable state there (and not be able to roll over to any side). The same applies for the maximum but inversely (imagine that the mountain is hollow and then flip it around).

This requires that at a given point, going towards any direction would result in a consist height sign change. Otherwise if the signs are opposite you encounter a **saddle point.**

Other critical points, that are max or min include "knikjes" (the function is not differentiable point).

Example: the cone $\mathbf{z = 4 - \sqrt{x^2 + y^2}}$



$$f_x(x, y) = -\frac{x}{\sqrt{x^2 + y^2}} \implies f_x(0, 0) \text{ not defined!}$$

$$f_y(x, y) = -\frac{y}{\sqrt{x^2 + y^2}} \implies f_y(0, 0) \text{ not defined!}$$

Definition of critical point

Definition:
A point (a, b) is a <u>critical point</u> of f if
- $f_x(a, b) = 0$ **and** $f_y(a, b) = 0$

or
- **if one of these partial derivatives does not exist in that point.**

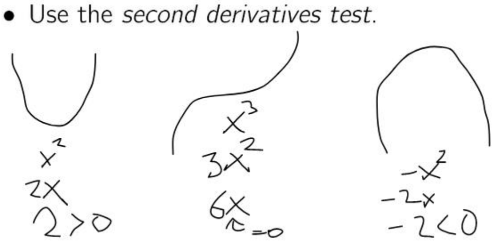
The second condition defining also a **singular point.**

Not every critical point gives you a maximum or minimum. But a maximum and minimum are by definition critical points.

Determine the type of critical point

Given a critical point, is it a local maximum, local minimum or a saddle point?

- Look at the contour diagram or at the graph;
- Use the *second derivatives test.*



(for 1 variable)

If f''(x) = 0 then it neither increases nor decreases.

Second Derivatives Test

Theorem:
Suppose that the second-order partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2.$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
- If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
- If $D < 0$, then (a, b) is a saddle point.

Given is the function $f(x,\ y) = y^2 - x^2y + 5x^2 - 6y$.

The function has three critical (= stationary) points (try to verify this yourself!).

Call these points P, Q and R.

Give point $P = (a,\ b)$ with $a = 0$. Enter the answer as a 1x2 matrix.

Yeah! That's right. The correct answer is $\begin{pmatrix} 0 & 3 \end{pmatrix}$.

To determine critical points: demand that $\nabla f(x, y) = \langle 0, 0 \rangle$. This gives:

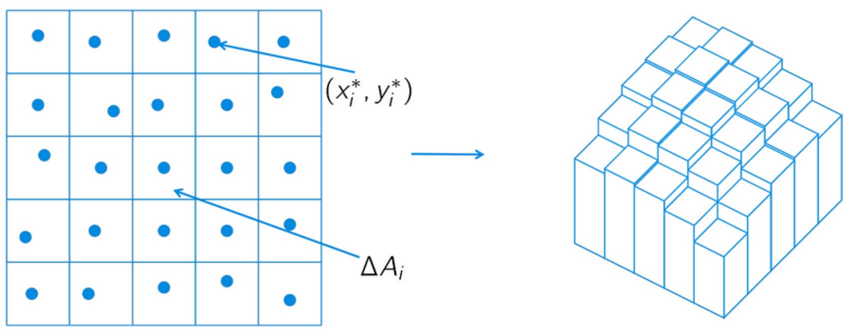
- $f_x(x,\ y) = -2xy + 2 \cdot 5x = 0$ (1)
- $f_y(x,\ y) = 2y - x^2 - 6 = 0$ (2)

From (1) it follows that $x = 0$ or $y = 5$.

Filling in $x = 0$ into (2) gives $y = \frac{6}{2}$.

Double Integrals over rectangles

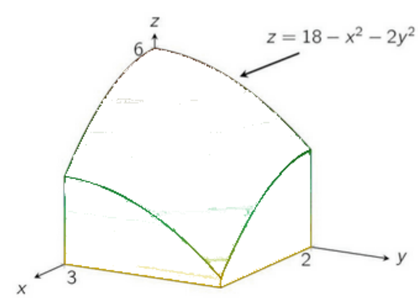
Riemann sum for double integrals



Definition:
Riemann sum = $\sum_{i=1}^{N} f(x_i^*, y_i^*)\Delta A_i$

Double integrals

In the case where $z = f(x, y)$ is **non-negative** on D: $\iint_D f(x, y)\, dA$ is the **volume** of the region above D and below the graph of f.



Definition of the (Riemann) integral $\iint_D f(x, y)\, dA$

Definition:
The integral of f over a region D is

$$\iint_D f(x, y)\, dA = \lim_{\Delta A_i \to 0^+} \sum_i f(x_i^*, y_i^*)\Delta A_i$$

if this limit exists.

Theorem:
The Riemann integral exists for any continous function on a well-behaved region D.

Rules of calculation

Let f, g be continuous functions on $D \subset \mathbb{R}^2$.

$$\iint_D f(x,y) + g(x,y)\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$$

If $f \geq g$ on D, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$

If D_1 and D_2 are disjoint, then

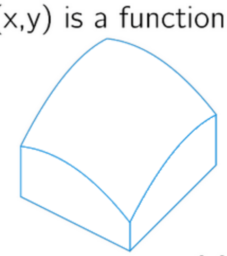
$$\iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA = \iint_{D_1 \cup D_2} f(x,y)\, dA$$

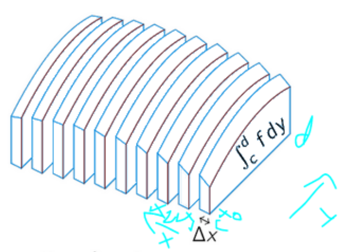
(The same as with single variable integrals)

Iterated integrals on a rectangular domain

From double integral to iterated integral

Suppose R is the rectangle $[a,b] \times [c,d]$ and f(x,y) is a function.





$$\iint_R f(x,y)\, dA \approx \sum_i (\text{Area slice}_i) \cdot \Delta x$$

$$\iint_R f(x,y)\, dA = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Theorem:
The double integral of $f(x,y)$ over a rectangle $R = [a,b] \times [c,d]$ is given by the iterated integrals:

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Exact evaluation: examples

$\iint_R (4-y)\,dA$ with $R = [0,2] \times [0,3]$

$\int_0^3\int_0^2 (4-y)\,dx\,dy = \int_0^3 (4-y)x\Big|_0^2\,dy = \int_0^3 8-2y\,dy = 8y-y^2\Big|_0^3$

$= 24 - 9 = 15$

$\iint_R (18 - x^2 - 2y^2)\,dA$ with $R = [0,3] \times [0,2]$

$\int_0^2\int_0^3 (18-x^2-2y^2)\,dx\,dy = \int_0^2 \left[18x - \frac{1}{3}x^3 - 2y^2x\right]_0^3 dy =$

$= \int_0^2 45 - 6y^2\,dy = \left[45y - 2y^3\right]_0^2 = 90 - 16 = 74$

Order of integration

Which order of integration would you choose for the following double integral?

$I = \iint_R y\sin(xy)\,dA, \quad R = [0,2] \times [0, \frac{1}{2}\pi]$

easies

A. $\int_0^{\frac{1}{2}\pi}\int_0^2 y\sin(xy)\,dx\,dy$

$a\sin(ax) \rightsquigarrow -\cos(ax)$

B. $\int_0^2\int_0^{\frac{1}{2}\pi} y\sin(xy)\,dy\,dx$

$y\sin(ay)$ use int by parts

Order of integration

Which order of integration would you choose for the following double integral? symmetric

$I = \iint_R \frac{x^3}{1+y^4}\,dA, \quad R = [-2,2] \times [0,2]$

$\frac{x^3}{1+a^4}$ ← odd

A. $\int_0^2\int_{-2}^2 \frac{x^3}{1+y^4}\,dx\,dy = 0$

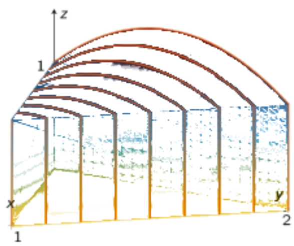
B. $\int_{-2}^2\int_0^2 \frac{x^3}{1+y^4}\,dy\,dx$

$\frac{a}{1+x^4}$ . . .

Area of symmetric odd function interval = 0

Double integrals over simple type I & II regions

The volume of the cake



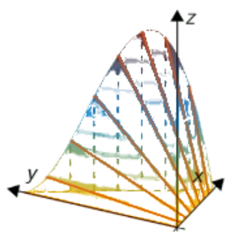
$$\text{Volume} = \sum_i \text{Vol}_{\text{slice}_i} = \sum_i \int f(x_i, y)\,dy\,\Delta x$$

$$\to \int \left(\int f(x,y)\,dy\right)\,dx$$

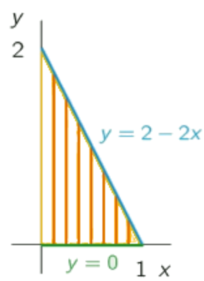
Constants

May depend on x

An example



$f(x,y) = 15x^2y$



$$\text{Area Slice} = \int_{y=0}^{2-2x} 15x^2y\,dy = \left[\tfrac{15}{2}x^2y^2\right]_{y=0}^{2-2x} = \tfrac{15}{2}x^2(2-2x)^2$$

$$\text{Volume} = \int_{x=0}^{1}\int_{y=0}^{2-2x} 15x^2y\,dy\,dx = \int_{x=0}^{1} \tfrac{15}{2}x^2(2-2x)^2\,dx = 1$$

The same applies to slicing the cake parallel on the x-axis

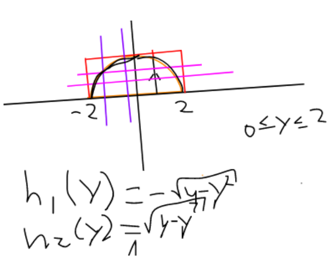
$$\text{Area Slice} = \int_{x=0}^{1-\frac{1}{2}y} 15x^2y\,dx = \left[5x^3y\right]_{x=0}^{1-\frac{1}{2}y} = 5(1-\tfrac{1}{2}y)^3y$$

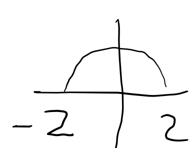
$$\text{Volume} = \int_{y=0}^{2}\int_{x=0}^{1-\frac{1}{2}y} 15x^2y\,dx\,dy = \int_{y=0}^{2} 5(1-\tfrac{1}{2}y)^3y\,dy = 1$$

Find the correct integral

Let D be the region between the lines $y = 0$ and $y = \sqrt{4-x^2}$.
Then $\iint_D y\,dA$ is equal to:

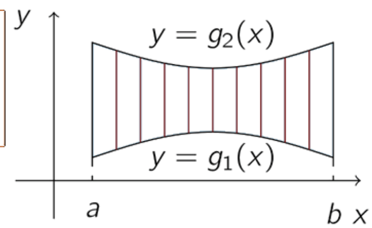
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} y\,dy\,dx$$



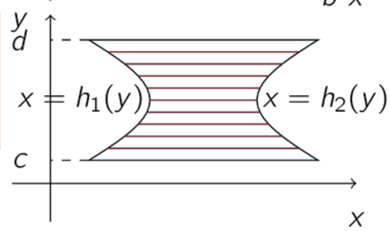


Regions of type I and/or type II

Definition:
A <u>region of type I</u> is a region of the form
$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$



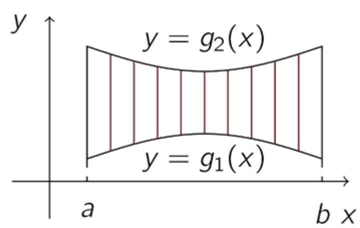
Definition:
A <u>region of type II</u> is a region of the form
$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$

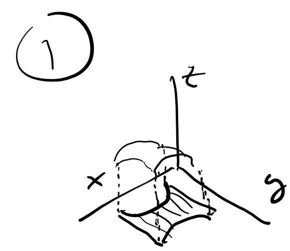


We cannot change the order of the double integrations. In fact the inside integral must output a function of x (so we derive on y) so that the outer integral can derive on x (see that the boundaries of the outer integral are constants whereas the inside integral has functions.

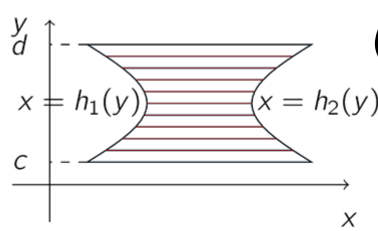
Regions of type I and/or type II

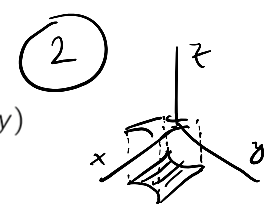
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$





$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$





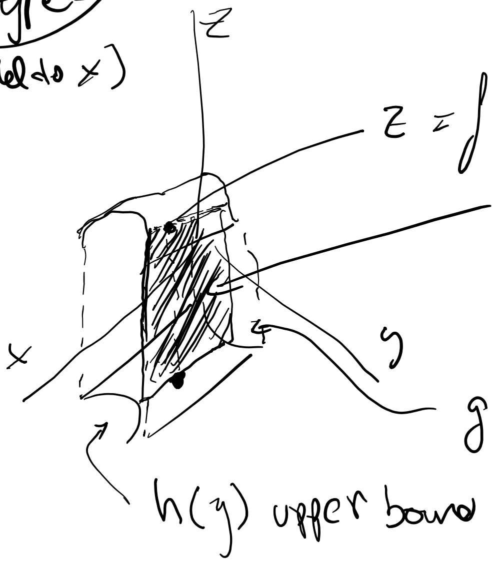
A region that is both type 1 and type 2 include rectangles, triangles, circles, pentagons…

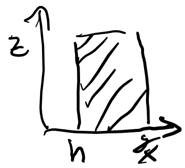
The region type is defined by whether we can draw verticals lines along the whole shape without lifting the pen and whether we can draw horizontal lines without lifting the pen.

Parallel to
- y • Vertical = type 1 = integrate y first
- x • Horizontal = type 2 = integrate x first

Type 2 is better suited when we are forced to invert f(x) or if h(y) is already given

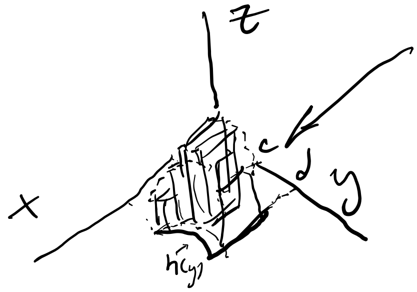
type 2 (parallel to x)

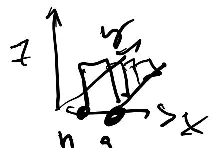




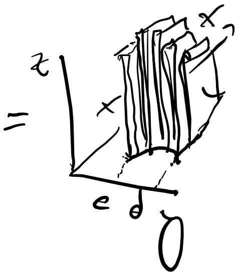
$= \int_{g(y)}^{h(y)} z \cdot dx = A(y)$

f(x,y) but y is used as a constant





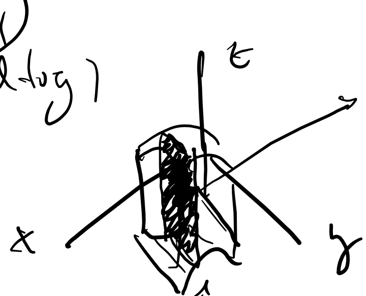
$=$



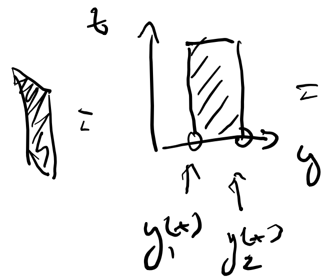
$= \int_d^c A(y) \cdot dy = V$

$$V = \int_e^d \left(\int_{g(y)}^{h(y)} f(x,y)\, dx \right) dy$$

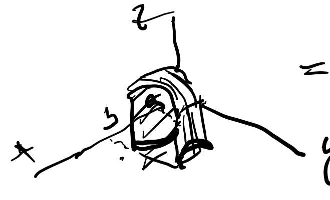
type 1 (parallel to y)



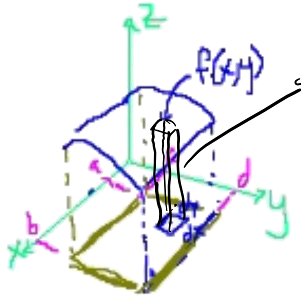
$y_2(x) = -(x-3)^2 + 5$



$= \int_{y_1(x)}^{y_2(x)} z \cdot dy = A(x)$



$= \int_a^b A(x) \cdot dx = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} z \cdot dy \right) dx$



$$\int f(x,y)\cdot dx\,dy = \int f(x,y)\cdot dA$$

→ (constant)

$$\iint_D f(x,y)\,dA = \int_{x\ lower}^{x\ upper}\left(\int_{y\ lower}^{y\ upper} z\cdot dy\right)\cdot dx$$

y upper → (in terms of x)

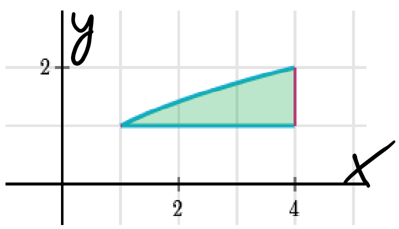
y lower → (in terms of x)

x lower → constant

$$\int_1^4 \int_1^{\sqrt{x}} dy\,dx$$

Switch the bounds of the double integral.

The first step whenever we want to switch bounds is to sketch the region of integration that we're given. Here, we see $1 < x < 4$ and $1 < y < \sqrt{x}$. Therefore:

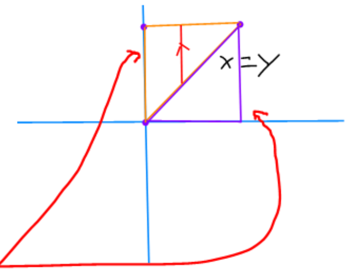


Because we're switching bounds to $dx\,dy$, we need to start with numeric bounds for y. We see that $1 < y < 2$. Then we can define x in terms of y. Thus, $y^2 < x < 4$.

In conclusion, the double integral after switching bounds is:

$$\int_1^2 \int_{y^2}^4 dx\,dy$$

Find the correct integral



Let R be the triangle with vertices $(0,0)$, $(0,2)$ and $(2,2)$. The double integral of a function f over R is equal to:

A. $\int_0^2 \int_0^x f(x,y)\,dy\,dx$

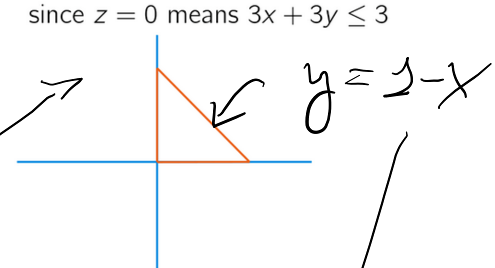
B. $\int_0^2 \int_x^2 f(x,y)\,dy\,dx$

Iterated integrals to calculate volume

Example

$$E : \begin{cases} x \geq 0,\ y \geq 0,\ z \geq 0 \\ 3x + 3y + z \leq 3 \end{cases}$$

since $z = 0$ means $3x + 3y \leq 3$



What is the volume of E?

Integrate the function $z = f(x,y) = 3 - 3x - 3y$ over the domain D.

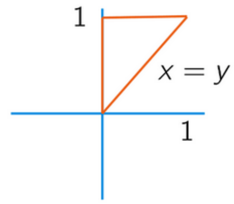
$$\mathrm{Vol}(E) = \iint_D (3 - 3x - 3y)\,dA$$

where $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$

$$\int_0^1 \int_0^{1-x} (3 - 3x - 3y)\,dy\,dx = \int_0^1 \left[3y - 3xy - \frac{3}{2}y^2\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{3}{2} - 3x + \frac{3}{2}x^2\,dx = \left[\frac{3}{2}x - \frac{3}{2}x^2 + \frac{1}{2}x^3\right]_0^1 = \frac{3}{2} - \frac{3}{2} + \frac{1}{2} = \frac{1}{2}$$

Reversing the order of integration



Example: $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$

$$\int_0^1 \int_0^y \cos(y^2)\,dx\,dy = \int_0^1 \left[x\cos(y^2)\right]_0^y dx\,dy = \int_0^1 y\cos(y^2)\,dy$$

$$= \int_0^1 \frac{1}{2}\cos(u)\,du = \left[-\frac{1}{2}\sin(u)\right]_0^1 = -\frac{\sin(1)}{2}$$

$u = y^2$

$$\iint_D 1\,dA = A(D)$$

11 If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\,dA \leqslant MA(D)$$

Exercises

Given is the integral $I = \int_0^6 \int_{\sqrt{\frac{x}{6}}}^1 e^{-y^3}\,dy\,dx.$

Hint: Change the order of integration.

Calculate the integral I.

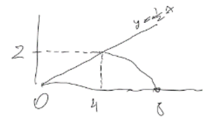
The correct answer is $I = -\frac{2}{e} + 2.$

Changing the order of integration, we find

$$I = \int_0^1 \int_0^{6y^2} e^{-y^3}\,dx\,dy = -\frac{2}{e} + 2.$$

Given is the integral $I = \int\int_D y\, dA$. Here D is the area in the first quadrant enclosed by the x-axis, the line $y = \frac{1}{2}x$, and the parabola $y = \sqrt{8-x}$.

Calculate the integral I.



$\frac{1}{2}x = \sqrt{8-x}$

$\frac{1}{4}x^2 = 8-x$

$x^2+4x-32=0$

$(x+8)(x-4)=0$

$x=4$

$0 < x < 8$

$0 < y < 2$

but it's not a square

let x then be bounded by

$y^2 = 8-x$

$2y < x < 8-y^2$

$z = y$ (by definition)

$$\int_0^2 \left(\int_{2y}^{8-y^2} y\, dx\right) dy = \int_0^2 \left(x\cdot y \Big|_{2y}^{8-y^2} \right) dy$$

$$= \int_0^2 (8y - y^3 - 2y^2)\, dy = 4y^2 - \frac{1}{4}y^4 - \frac{2}{3}y^3 \Big|_0^2$$

$$= 16 - \frac{16}{4} - \frac{16}{3} = 16\left(1 - \frac{1}{4} - \frac{1}{3}\right)$$

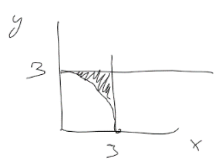
$$= 16\left(\frac{12-3-4}{12}\right) = \frac{8\cdot 5}{6} = \frac{40}{6} = \frac{20}{3}$$

Let G be the solid (three-dimensional region) enclosed by the cylinder $z = \sqrt{9-x^2}$ and the region $0 \le x \le \sqrt{9-y^2}$, $0 \le y \le 3$. Sketch the three-dimensional solid G and find the volume of solid G.

Hint: choose a useful order of integration.

Determine $\mathrm{vol}(G)$.

$z = \sqrt{9-x^2}$



$x = \sqrt{9-y^2}$

$x^2 = 9-y^2$

$x^2 - 9 = -y^2$

$9 - x^2 = y^2$

$y = \sqrt{9-x^2}$

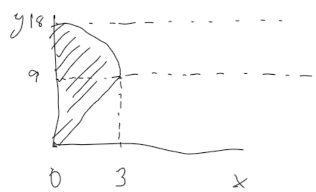
$$\int_0^3 \int_0^{\sqrt{9-y^2}} z\, dx\, dy$$

$$= \int_0^3 \left(x\cdot z \Big|^{\sqrt{9-y^2}} \right) dy = \int_0^3 \sqrt{9-y^2}\,\sqrt{9-y^2}\, dy = \int_0^3 9 - y^2\, dy = 9y - \frac{1}{3}y^3 \Big|_0^3$$

$$= 27 - 9 = 18$$

Consider the integral $I = \int_0^9 \int_0^{\frac{y}{3}} f(x,\ y)\ dx\ dy + \int_9^{18} \int_0^{\sqrt{18-y}} f(x,\ y)\ dx\ dy$. Sketch the region of integration. Then change the order of integration and determine the corresponding boundaries. $I = \int_c^d \int_a^b f(x, y)\ dy\ dx$

Determine a. b, c, d



$x = \frac{y}{3}$

$3x = y$

$x = \sqrt{18-y}$

$x^2 = 18-y$

$y = 18-x^2$

$3x = 18 - x^2$

$-x^2 + 18 - 3x = 0$

$x^2 + 3x - 18 = 0$

$(x+6)(x-3) = 0$

$x = 3$

$0 < x < 3$

inv $< y <$ inv

determine boundaries: use $x=2$

$3 \cdot 2 = 6 \quad 18 - 2^2 = 14$

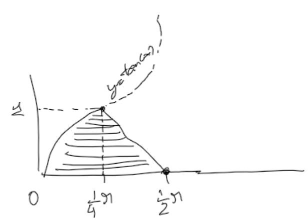
So:

$3x < y < 18 - x^2$

$$\int_0^3 \left(\int_{3x}^{18-x^2} z \cdot dy \right) dx$$

Consider the integral $I = \int_0^{\frac{\pi}{4}} \int_0^{\tan(x)} f(x,\ y)\ dy\ dx + \int_{\frac{\pi}{4}}^{\frac{2\pi}{4}} \int_0^{\frac{2}{1} - \frac{4}{1\pi} \cdot x} f(x,\ y)\ dy\ dx$. Sketch the integration region. Then change the order of integration and determine the corresponding boundaries.

$I = \int_c^d \int_a^b f(x,\ y)\ dx\ dy$



edges

$y = 2 - \frac{4}{\pi} x$

$0 = 2 - \frac{4}{\pi} x$

$\frac{4}{\pi} x = 2$

$x = \frac{2\pi}{4} = \frac{1}{2}\pi$

$\tan(\frac{1}{4}\pi) = 1$

$2 - \frac{4}{\pi} \cdot \frac{1}{4}\pi = 2 - 1 = 1$

$y = 2 - \frac{4}{\pi} x$

$-y + 2 = \frac{4}{\pi} x$

$\frac{\pi}{4}(2-y) = x$

use $y = 0$

$\frac{\pi}{4} \cdot 2 = \frac{1}{2}\pi$

(upper x-bound)

$y = \tan(x)$

$\arctan(y) = \arctan(\tan(x))$

$x = \arctan(y)$

$0 < y < 1$

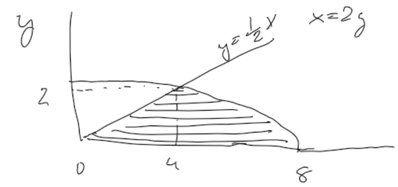
$x(y) < x < x(y)$

$\arctan(y) < x < \frac{1}{4}\pi(2-y)$

$$\int_0^1 \left(\int_{\arctan(y)}^{\frac{1}{4}\pi(2-y)} z \cdot dx \right) dy$$

Given is the integral $I = \int\int_D y\, dA$. Here D is the area in the first quadrant enclosed by the x-axis, the line $y = \frac{1}{2}x$, and the parabola $y = \sqrt{8-x}$.

Calculate the integral I.



$y = \sqrt{8-x}$

$-y^2 = -8+x$

$x = 8-y^2$

(easier to integrate)

$0 < y < 2$ ← use $y=1$: $x=2$ vs $x=8-1=7$

$2y < x < 8-y^2$

$8-y^2 = 2y$ → $y=2$; $2=\frac{1}{2}x$ → $x=4$

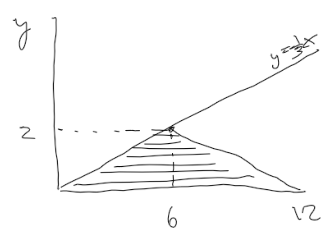
$$\int_0^2 \int_{2y}^{8-y^2} y \cdot dx \cdot dy = \int_0^2 \left(x \cdot y \Big|_{2y}^{8-y^2} \right) \cdot dy = \int_0^2 \left(8y - y^3 - 2y^2\right) \cdot dy$$

$$= \frac{8}{2}y^2 - \frac{1}{4}y^4 - \frac{2}{3}y^3 \Big|_0^2 = 16 - 4 - \frac{2}{3} \cdot 8 = 12 - \frac{16}{3}$$

$$= \frac{36-16}{3} = \frac{20}{3}$$

Given is the integral $I = \iint_D y\, dA$. Here, D is the area in the first quadrant enclosed by the x-axis and the lines $y = \frac{1}{3}x$ and $y = -\frac{1}{3}(x-6)+2$.

Calculate the integral I.



$0 = -\frac{1}{3}(x-6)+2$

$x-6-6=0$

$x=12$

$y = -\frac{1}{3}x + 4$

$0 < x < 12$ ← big number, and this has fractions...

better to integrate x first.

$x = 3y$ | $-3y = x-6-6$

$x = 12-3y$

$0 < y < 2$

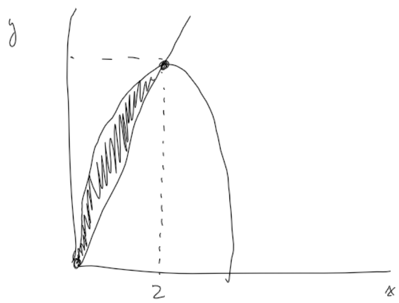
$3y < x < 12-3y$

$$\int_0^2 \int_{3y}^{12-3y} y \cdot dx \cdot dy = \int_0^2 \left(x \cdot y \Big|_{3y}^{12-3y} \right) dy = \int_0^2 \left(12y - 6y^2\right) dy$$

$$= \frac{12}{2}y^2 - \frac{6}{3}y^3 \Big|_0^2 = 6 \cdot 4 - 2 \cdot 8 = 24 - 16 = 8$$

Given is the integral $I = \iint_D x \, dA$. Here D is the region in the first quadrant above the line $y = 4x$ and under the parabola $y = 2x(4 - x)$.

Calculate the integral I.



$4x = 2x(4-x)$, $2 = 4-x$, $x=2$ | $y = 2x(4-x) = 8x - x^2 = -x^2 + 8x$

$0 < x < 2 \mid 4x < y < -x^2 + 8x$

$$\int_0^2 \left(\int_{4x}^{-x^2+8x} x \cdot dy \right) dx = \int_0^2 \left(y \cdot x \Big|_{4x}^{-x^2+8x} \right) \cdot dx = \int_0^2 \left(-2x^3 + 4x^2 \right) dx$$

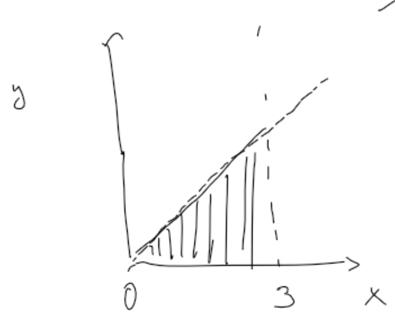
$$= -\frac{2}{4}x^4 + \frac{4}{3}x^3 \Big|_0^2 = -2 \cdot \frac{16}{4} + \frac{4 \cdot 8}{3} = -8 + 8 \cdot \frac{4}{3}$$

$$= 8\left(\frac{4}{3} - 1\right) = 8\left(\frac{1}{3}\right) = \frac{8}{3}$$

Let G be the solid (three-dimensional region) enclosed by the parabolic cylinder $z = 9 - x^2$ and the region $0 \le x \le 3, 0 \le y \le x$. Sketch the three-dimensional region (solid) G and find the volume of solid G.

Determine $\text{vol}(G)$.

$z = 9 - x^2$



$$\int_0^3 \int_0^x (9 - x^2) \cdot dy \, dx$$

$$= \int_0^3 y \cdot (9 - x^2) \Big|_0^x \cdot dx$$

$$= \int_0^3 9x - x^3 \, dx = \frac{9}{2}x^2 - \frac{1}{4}x^4 \Big|_0^3$$

$$= \frac{9}{2} \cdot 3^2 - \frac{1}{4} \cdot 3^4 = \frac{81 \cdot 2}{4} - \frac{81}{4} = \frac{81}{4}$$

Let D be the area in the first quadrant enclosed by the coordinate-axes and $y = 1 - x^2$,

and let $f(x, y) = 2x\cos(2y)$ be the given function.

Let f_m be the *average* of the function over area D.

Then the following holds:

The mean of a function $f(x, y)$ over an area D is by definition equal to:

$$f_m = \frac{1}{\text{area}(D)} \iint_D f(x,y)\,dA$$

Determine f_m.

$$\int_0^1 1-x^2\,dx = x - \frac{1}{3}x^3 \Big|_0^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$$\int_0^1 \left(\int_0^{1-x^2} 2x \cdot \cos(2y) \cdot dy \right) dx = \int_0^1 \left(x \cdot \sin(2y) \Big|_0^{1-x^2} \right) dx$$

$$= \int_0^1 \left(x \sin(2-2x^2) \right) dx$$

let $u = 2 - 2x^2$

$$\frac{du}{dx} = -4x$$

$$-4x\,dx = du$$

$$dx = \frac{du}{-4x}$$

$$= \int_{u(0)}^{u(1)} x \cdot \sin(u) \cdot \frac{du}{-4x}$$

$$= \int_2^0 -\frac{1}{4}\sin(u) \cdot du$$

$u(1) = 2 - 2 = 0$

$u(0) = 2 - 0 = 2$

$$= \frac{1}{4}\cos(u) \Big|_2^0 = \frac{1}{4}\cos(0) - \frac{1}{4}\cos(2)$$

$$= \frac{1}{4} - \frac{1}{4}\cos(2) = \frac{1}{4}(1 - \cos(2))$$

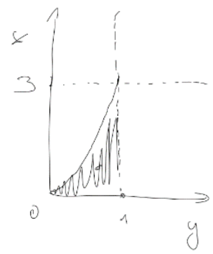
$$f_m = \frac{1}{\frac{2}{3}} \cdot \frac{1}{4}(1 - \cos(2))$$

$$= \frac{3}{8}(1 - \cos(2))$$

Given is the integral $I = \int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{-y^3}\, dy\, dx$.

Hint: Change the order of integration.

Calculate the integral I.



$y = \sqrt{\frac{x}{3}}$ $\quad 3y^2 = 3$

$y^2 = \frac{x}{3}$

$x = 3y^2$

$$\int_0^1 \int_0^{3y^2} e^{-y^3} dx\, dy = \int_0^1 \left(x \cdot e^{-y^3} \Big|_0^{3y^2} \right) dy = \int_0^1 \left(3y^2 e^{-y^3} \right) dy$$

let $u = -y^3$, $\frac{du}{dy} = -3y^2$, $\frac{du}{-3y^2} = dy$

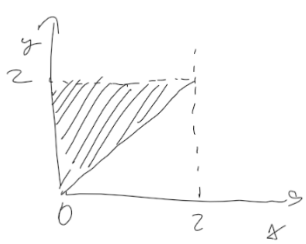
$$= \int_{u(0)}^{u(1)} 3y^2 e^{u} \frac{du}{-3y^2} = \int_0^{-1} -e^{u} \cdot du = -e^{u} \Big|_0^{-1}$$

$u(1) = -1$, $u(0) = 0$

$$= -e^{-1} - (-e^{0}) = -\frac{1}{e} + 1$$

The iterated integral $\int_0^2 \int_x^2 e^{\frac{x}{y}} dy\, dx$ is equal to

$2(e-1)$	e	$\frac{1}{4}e$	$\frac{1}{2}e$	$\frac{1}{2}(e-1)$	$2e$	$\frac{1}{4}(e-1)$	$e-1$



$0 < y < 2$, $0 < x < y$

$$\int_0^2 \left(\int_0^y e^{\frac{1}{y} \cdot x} \cdot dx \right) dy$$

$$= \int_0^2 \left(y \cdot e^{\frac{x}{y}} \Big|_0^y \right) dy = \int_0^2 \left(y e^{1} - y e^{0} \right) dy = \int_0^2 y(e-1) \cdot dy$$

watch out! $e^0 = 1$

$$= \frac{1}{2} y^2 (e-1) \Big|_0^2 = \frac{1}{2} \cdot 4 (e-1) = 2(e-1)$$

Week 9. Complex numbers

The same way that we use x as the variable for real numbers and k for integers (sometimes i and n), for complex numbers we use z.

Complex numbers

$z = Re(z) + i\,Im(z)$

Introduce a symbol i such that $\boxed{i^2 = -1}$.

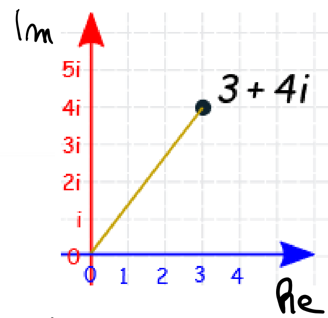
Definition

A complex number is a number of the form

$$z = a + bi$$

where a and b are real.

- real part: $\text{Re}(z) = a$,
- imaginary part: $\text{Im}(z) = b$.



All real numbers are complex numbers with imaginary part equal to zero.

Operations

Suppose $z = a + bi$ and $w = c + di$ are complex numbers. Then the following are again complex numbers:

- $z + w$
- $z - w$
- $z\,w$
- $\frac{z}{w}$ (if $w \neq 0$)
- $\bar{z} = a - bi$ (complex conjugate)

Addition of complex numbers z and w

Re(z+w) = Re(z) + Re(w)

Im(z+w) = Im(z) + Im(w)

Multiplication of z and w

Re(zw) = Re(z)Re(w) – Im(z)Im(w)

Im(zw) = Re(z)Im(w) + Im(z)Re(w)

Addition of z and !z

Re(z) = ½(z+!z)

Subtraction of z and !z

Im(z) = ½(z-!z)

Properties of Conjugates

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

$$z\bar{z} = |z|^2$$

Division of complex numbers

Division with square roots in the denominator:

$$\frac{1}{2-\sqrt{2}} = \frac{1}{2-\sqrt{2}}\frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{2+\sqrt{2}}{2^2-(\sqrt{2})^2}$$

$$= \frac{2+\sqrt{2}}{2} = 1 + \frac{1}{2}\sqrt{2}$$

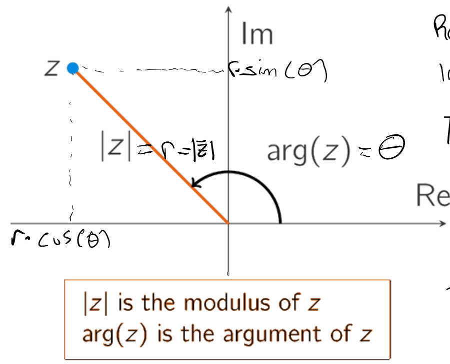
Division with complex numbers in the denominator:

$$\frac{2-3i}{1+2i} = \frac{2-3i}{1+2i}\frac{1-2i}{1-2i} = \frac{2-4i-3i+6i^2}{1^2-(2i)^2}$$

$$= \frac{-4-7i}{5} = -\frac{4}{5} - \frac{7}{5}i$$

Polar form

Complex numbers: geometry



$Re(z) = r \cdot \cos(\arg(z))$

$Im(z) = r \cdot \sin(\arg(z))$

$|z| = \sqrt{Re^2(z) + Im^2(z)} = \sqrt{z \cdot \bar{z}}$

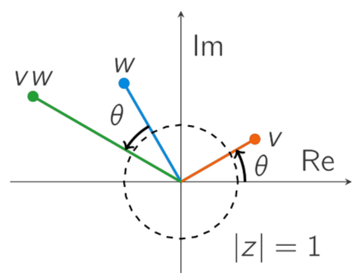
$\tan(\theta) = \frac{b}{a}$

Polar form of z:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Polar form is suitable for multiplication and division (especially long i.e. powers) but harder for addition.

- $|v\,w| = |v|\,|w|$
- $\arg(v\,w) = \arg(v) + \arg(w)$



$e^{it} = \cos(t) + i\sin(t)$ → **Euler's Identity**

Suppose

$$z = r\,(\cos(\theta) + i\sin(\theta)) = r\,e^{i\theta}$$
$$w = s\,(\cos(\phi) + i\sin(\phi)) = s\,e^{i\phi}$$

Then:

$$zw = rs\,(\cos(\theta+\phi) + i\sin(\theta+\phi)) = rs\,e^{i(\theta+\phi)}$$
$$\frac{z}{w} = \frac{r}{s}\,(\cos(\theta-\phi) + i\sin(\theta-\phi)) = \frac{r}{s}\,e^{i(\theta-\phi)} \quad (\text{for } s \neq 0)$$
$$z^n = r^n\,(\cos(n\theta) + i\sin(n\theta)) = r^n\,e^{in\theta}$$

3 Roots of a Complex Number Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

Binomial equations

Exercise: $z^4 = -16$

$|z|^4 = |-16| = 16 \rightarrow |z| = 2$

$4\arg(z) = \arg(-16) = \pi + 2k\pi$ k integer

$\arg(z) = \frac{\pi}{4} + \frac{k\pi}{2} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\sqrt{2} + i\sqrt{2}, -\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}, \sqrt{2} - i\sqrt{2}$

Solution:

Just use k from 0 to n-1

Polynomial equations

$6z^5 + (3+i)z^3 + 2z^2 - i = c(z - a_1)(z - a_2) \cdots (z - a_5)$

Theorem (Fundamental Theorem of Algebra)
Let $p(z)$ be a polynomial in z of degree n, $n \geq 1$.
Then we can write $p(z) = c(z - a_1)(z - a_2) \ldots (z - a_n)$,
$c, a_1, \ldots, a_n \in \mathbb{C}$, $c \neq 0$.

$a_1, \ldots, a_n$ are the roots of $p(z)$.

There are as many solutions as the power of the complex number polynomial. But those solutions might not necessarily be distinct.

Theorem
Let p be a polynomial with real coefficients.
If $p(z) = 0$, then $p(\overline{z}) = 0$ as well.

z represents a solution (and every solution has its conjugate as validation solution too)

Exercises

Give the imaginary part of

$$\frac{1 + 2I}{1 - 2I}.$$

✗ $2I$	✗ 2	✓ $\frac{4}{5}$	✗ $\frac{4}{5}I$

$|\overline{z}| = |z| = |re^{i\theta}| = r$

$\left|\frac{1}{z}\right| = \frac{1}{r}$ $\quad |-z| = |z| = r$

$|zw| = r_1 \cdot r_2$

$\left|\frac{z}{w}\right| = \frac{r_1}{r_2}$

$\arg(\overline{z}) = -\arg(z) = -t$

$\arg\left(\frac{1}{z}\right) = -t$

$\arg(-z) = \arg(z) + \pi$